THE DIRICHLET INVERSE OF A DIGIT DESIGN AND THE TRANSPORT OF ZEROS

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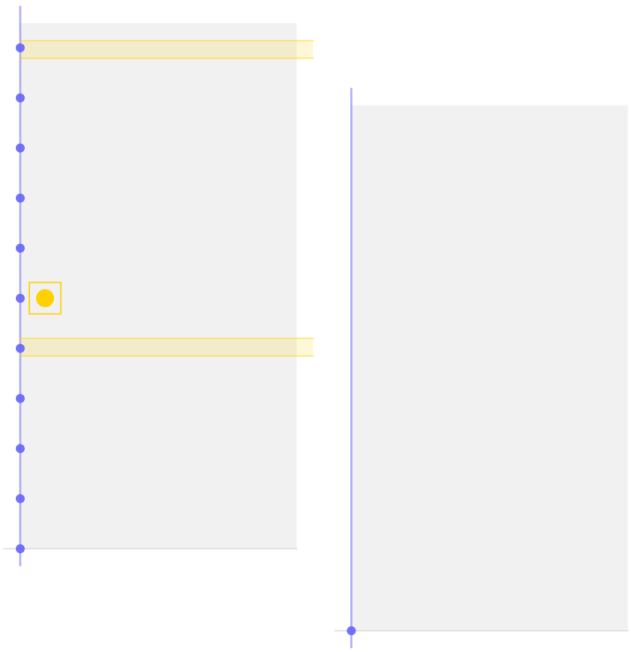
MrlyProd, Inc.
First published 2026-09-07, revised 2026-09-08



Abstract. Cross out some digits of a base and keep the numbers you can still write. That set has a zeta function, a Dirichlet series built from the set alone, and this paper asks what plays the part of the Mobius function for it. The answer is forced: the indicator of the set has a Dirichlet inverse, and it is the Mobius function only when no digit was crossed out. The inverse is worse behaved than the set it inverts. Each zero of the set's zeta function lying to the right of that series' own abscissa of convergence pushes the abscissa of the inverse at least that far right, so the partial sums of the inverse cannot be small. Two such zeros have been located inside certified boxes, one for base three keeping the digits zero and one, one for base ten with the digit nine removed, and in the second case the box lies to the right of the line where the count of all integers sits. The set's own Mobius function therefore does not cancel against the set's own size: it anti-cancels.

1. Introduction

Fix a base q and a set F of digits inside $\{0, 1, \dots, q-1\}$. The numbers you can still write using only those digits form a set: in base 10 with the digit 9 crossed out these are $1, \dots, 8, 10, \dots, 18, 20, \dots$, and there are 9^L of them below 10^L. Call such a set a *design* and write it S_F. Designs are the standard test bed for analytic number theory on a thin set; Maynard proved that infinitely many of them are prime in base 10 [9].

Every set of positive integers has a Dirichlet series, and the design's is

$$\zeta_F(s) \;=\; \sum_{n \in S_F} n^{-s},$$

which converges exactly in the half-plane $\operatorname{Re} s > \alpha$, where $\alpha = \log_q k$ and k is the number of surviving digits. When no digit is crossed out, S_F is all of the positive integers, $\alpha = 1$, and ζ_F is Riemann's zeta function.

In that classical case there is a second series, and it is the reason the subject exists. The Mobius function μ satisfies $\zeta(s)\sum_n \mu(n)n^{-s} = 1$: it is exactly the weight you must put on the integers to invert their zeta function. This paper asks the same question of a design. What must you weight S_F by to invert ζ_F, and how does that weight behave?

The first half of the answer is short and classical in shape. Any arithmetic function f with $f(1) = 1$ has a unique Dirichlet inverse, built by one recursion, and the indicator of S_F has $f(1) = 1$ as soon as the digit 1 survives. So there is a function ν_F, and only one, with

$$\zeta_F(s)N_F(s) = 1, \qquad N_F(s) = \sum_{n \geq 1} \nu_F(n) n^{-s},$$

and at the full digit set ν_F is μ on the nose. Nothing here is new; it is the standard construction, recalled in Theorem 3.8 to fix notation and to record where ν_F lives. What is worth saying is that ν_F is not μ cut down to the design, and it is not supported on the design either: it spills onto the multiplicative semigroup that S_F generates, and not onto all of that.

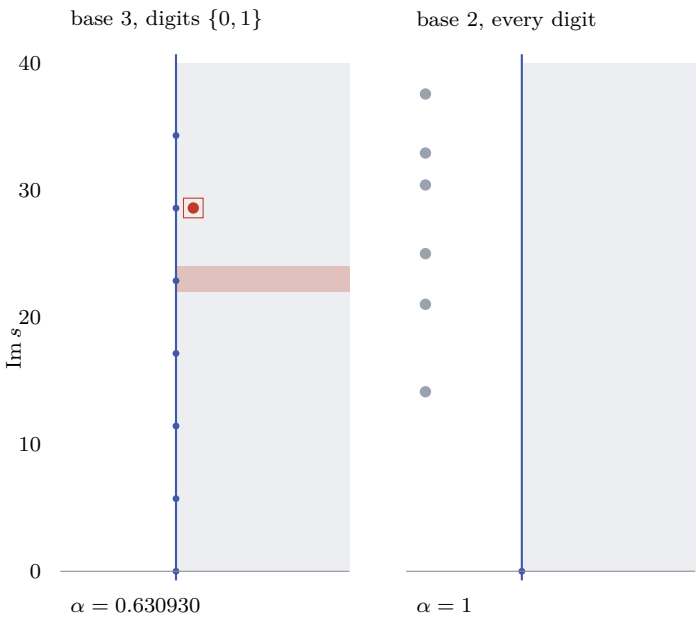


FIGURE 1. Two Dirichlet series with nonnegative coefficients, each drawn beside its own abscissa of convergence, out to height 40; the shaded half of each panel is that series' own region of absolute convergence. On the right, the integers: ζ has abscissa $\operatorname{Re} s = 1$ and, by the Euler product, not one zero in the shaded half. Its six zeros below height 40 are the grey points, all on $\operatorname{Re} s = 1/2$, all outside the shading. On the left, the design of base 3 on the digits $\{0, 1\}$: its abscissa is $\operatorname{Re} s = \alpha = 0.630930$, the poles of its continuation sit on that line in a lattice of period $2\pi/\log 3$, and two zeros lie inside the shaded half below height 40, with a third near height 57. The filled point is the one located inside a certified winding box, at $\rho = 0.720790 + 28.605680\,i$, with the box drawn around it. The strip is the other zero, whose height is censused but whose real part is not pinned, so it is drawn open. Every such zero is transported by Theorem 3.10 into a lower bound on the design's own Möbius partial sums.

The second half is where the design and the integers part company. For the integers, the Euler product $\zeta(s) = \prod_p (1-p^{-s})^{-1}$ forbids a zero anywhere in $\operatorname{Re} s > 1$, and that is why $1/\zeta$ converges as far to the right as ζ does. A design has no such product. Its indicator is multiplicative only when no digit was crossed out (Theorem 3.1), so the Euler route is not narrowed, it is absent, and nothing forbids ζ_F from vanishing inside its own half-plane of absolute convergence. It does vanish there. Fig. 1 is that statement drawn.

The consequence is one page of elementary complex analysis and is the point of the paper.

Theorem 1.1 (The transport, Theorem 3.10)**.** *Let $1 \in F$ and let ρ be a zero of ζ_F with $\operatorname{Re}\rho > \alpha$. Then $\sigma_c(N_F) \geq \operatorname{Re}\rho$, and hence*

$$\sum_{n\leq x} \nu_F(n) \;\neq\; O\Big(x^{\operatorname{Re}\rho-\varepsilon}\Big) \quad \text{for every } \varepsilon > 0.$$

The proof is the identity theorem on a connected half-plane and nothing more; it is stated as a theorem because of what it is fed, not because it is deep. What it is fed are two zeros located by the argument principle inside boxes of width 10^{-4} (Fact 3.13). At base 3 on the digits $\{0,1\}$ the box sits at $\operatorname{Re} s \in [0.72074, 0.72084]$, while the whole design has only $A_F(x) \asymp x^{0.630930}$ elements below x. So the design's own Mobius function does not cancel against the design's own mass: its partial sums are, along a sequence, larger than the number of terms being summed. At base 10 with the digit 9 removed the box sits at $\operatorname{Re} s \in [1.00150, 1.00168]$, to the right of $\operatorname{Re} s = 1$, so the partial sums exceed even the count of all integers below x. Both statements are about the limit superior; the finite census is far below both, and Fact 3.15 says by how much.

This is the observation the paper is for. Everything around it is either classical, elementary, or a measurement, and is labelled as such. In particular the continuation of ζ_F to the whole plane, its pole lattice, and the fact that all of its poles are simple and lie among $\alpha - m + 2\pi i j/\log q$ are Burnol's [5]; the engine that locates the zeros uses his Proposition 4.1 in a peeled form and claims nothing about it. That μ is not q-automatic, so that no automatic-sequence machinery continues a Mobius-weighted design series, is Coons [6]. The two other multiplicative structures a design carries, a Lyndon word product and a Beurling system on the primes inside S_F, are laid out in Section 3.7 exactly to say what each one cannot see, and the Beurling comparison is Beurling's [4] with the cautionary example of Diamond, Montgomery and Vorhauer [7].

Two words on what is not claimed. The transport is one-sided: it bounds $\sigma_c(N_F)$ from below and never from above, so the sentence "the rightmost zero is the abscissa" appears nowhere below. And the zero census is resolved rather than certified: the argument principle is run on adaptively refined contours with the largest surviving phase step printed, but nothing here bounds ζ_F'/ζ_F along a contour, so a pair of zeros closer together than the surviving spacing would be invisible to it. The two boxes of Fact 3.13 are the exception and carry a winding number of 1 each.

2. The design, its zeta and its inverse

Throughout, $q \geq 2$ is an integer base and $F \subseteq \{0, 1, \ldots, q-1\}$ a set of digits with $k = \#F \geq 1$. Write $e(x) = e^{2\pi i x}$.

Definition 2.1 (The design)**.** S_F is the set of positive integers every base-q digit of which lies in F, and $A_F(x) = \#\{n \le x : n \in S_F\}$ is its counting function. Write D_L for the set of length-L digit strings over F read as base-q integers, so $\#D_L = k^L$; if $0 \in F$ then D_L is the set of elements of $S_F \cup \{0\}$ below q^L and the D_L nest, and if $0 \notin F$ they partition S_F by length. The design is *full* when $F = \{0, \dots, q-1\}$, in which case S_F is all of $\mathbf{Z}_{>0}$.

Definition 2.2 (The design zeta and the abscissa)**.** $\zeta_F(s) = \sum_{n \in S_F} n^{-s}$ and $\alpha = \log_q k$. Since $A_F(q^L) \le k^L$ with equality up to one term, the abscissa of absolute convergence of ζ_F is exactly α, and by Landau's theorem on Dirichlet series with nonnegative coefficients ζ_F is singular at $s = \alpha$. Burnol [5] continues ζ_F meromorphically to $\mathbf{C}$: all poles are simple and lie among the lattice points $s_{m,j} = \alpha - m + 2\pi i j / \log q$ with $m \ge 0$ and $j \in \mathbf{Z}$, the pole at $s_{0,0} = \alpha$ has positive residue, and $\prod_{m \ge 0}(1 - kq^{-s-m})\zeta_F(s)$ is entire. We write $Z(s) = \zeta_F(s)(1 - kq^{-s})$ for the *Lyndon cofactor*, which by the same recursion is analytic on $\operatorname{Re} s > \alpha - 1$.

Definition 2.3 (The design Mobius series and the inverse)**.** $M_F(s) = \sum_{n \in S_F} \mu(n) n^{-s}$ is the design Mobius series, the classical μ read only on the design. It is dominated termwise by ζ_F and so converges absolutely on $\operatorname{Re} s > \alpha$. Separately, when $1 \in F$, ν_F denotes the Dirichlet inverse of the indicator $\mathbf{1}_{S_F}$ and $N_F(s) = \sum_{n \ge 1} \nu_F(n) n^{-s}$ its series, with $\sigma_c(N_F)$ and $\sigma_a(N_F)$ the abscissas of conditional and absolute convergence. These are two different objects and the paper keeps them apart: M_F carries the classical arithmetic onto the design, N_F is the design's own Mobius function.

Definition 2.4 (The position product)**.** $G_L(t) = \prod_{i<L} \sum_{d \in F} e(dq^i t)$, a trigonometric polynomial of degree below q^L, and for an arithmetic function a, $A(s,t) = \sum_{n \ge 1} a(n) e(-nt) n^{-s}$.

3. Results

Results are dressed by what stands behind them. A *Theorem* or *Lemma* has a complete proof in Section 4. A *Fact* is a finite computation, stated with its exact domain and with the script that produced it named; nothing unproved is dressed as a theorem. Every generator named below lives in the public research tree at `https://github.com/mrlyprod/mrlyprod` under `research/lab/`, and Section 5 says which of these rows the script shipped with this paper reruns and which it does not.

3.1. No Euler product over the primes.

Theorem 3.1 (The wall)**.** *Let $q \ge 2$ and $F \subseteq \{0, \dots, q-1\}$. The indicator $\mathbf{1}_{S_F}$ is multiplicative if and only if $F = \{0, \dots, q-1\}$.*

The proof in Section 4.1 is constructive: for every proper F containing 1 it writes down a coprime pair witnessing the failure, in three cases. If some digit $c \ge 2$ is missing, take the least such and use the repunits R_c and R_{c+1},

whose product carries no carry and whose digit set is exactly $\{1, \dots, c\}$. If only the digit 0 is missing then odd q uses $(2, (q^2+1)/2)$ and even q uses (q^2-1, q^2+1).

Fact 3.2 (The wall, swept)**.** Over all 8177 nonempty digit sets with $2 \le q \le 12$: 4083 have $1 \notin F$ and fail at $f(1) = 1$ alone, 11 are full, and the constructed witness of Theorem 3.1 is asserted at each of the remaining 4083, split 4072 repunit, 5 odd base, 6 even base. An independent minimal search over products up to 4000 finds a witness for every one of the 4083; the hardest is $q = 12$, $F = \{1\}$, where the least witness is the pair $(5, 377)$ with product 1885, against the constructed pair $(13, 157)$. Script of record: `lab/mrly-euler`, verb `wall`.

Consequently no design outside the full digit set carries an Euler product over the primes, and the classical bridge from ζ to $1/\zeta$ is not narrowed on a design but absent. The pair $\zeta_F M_F$ measures the damage exactly.

Theorem 3.3 (The pair comes apart)**.** *Let F be strictly inside $\{0, \dots, q-1\}$. Then at least one of the following holds: (i) $1 \notin F$, and the constant coefficient of $\zeta_F M_F$ is 0; (ii) some prime $p \in S_F$ has $p^2 \notin S_F$, and the coefficient of $\zeta_F M_F$ at p^2 is -1; (iii) $\zeta_F M_F = 1$, and then the least element $g > 1$ of S_F satisfies $\mu(g) = -1$ and every power g^j lies in S_F. At the full digit set $\zeta_F = \zeta$, $M_F = 1/\zeta$ and the two are inverse.*

Clause (iii) is a necessary condition on a hypothetical escapee, not a contradiction, so the theorem is a disjunction and the paper does not upgrade it to a universal. What it says is that an escaping design would have to be squarefree at its smallest nontrivial element and carry that element's whole geometric progression. In practice the pair comes apart at once.

Fact 3.4 (Where the pair breaks)**.** Over 257 digit sets ($q \le 8$ with $1 \in F$, plus base 10 missing 9, base 10 missing 0, and the full sets) the least $n > 1$ at which $\zeta_F M_F$ has a nonzero coefficient is at most 50. It is $n = 4$ for base 3 on $\{0, 1\}$, $n = 9$ for base 10 missing 9, and $n = 10$ for base 10 missing 0 with coefficient -2; the eight full sets in the list have no nonzero coefficient below $n = 4000$. Script of record: `lab/mrly-euler`, verb `pair`.

3.2. **The product that survives.** A design does have an exact Euler product. It is not indexed by primes but by digit positions, and it lives in the frequency variable.

Theorem 3.5 (The position product)**.** *For every integer n,*

$$\int_0^1 G_L(t) e(-nt)\, dt \;=\; \mathbf{1}_{D_L}(n),$$

and hence for every Dirichlet series $\sum_{n \ge 1} a(n) n^{-s}$ absolutely convergent at s,

$$\sum_{n \in D_L,\, n \ge 1} a(n) n^{-s} \;=\; \int_0^1 G_L(t) A(s,t)\, dt.$$

Taking $a = 1$ makes $A(s,t)$ the periodic zeta function $F(-t,s)$ of DLMF 25.13.1 [8], and taking $a = \mu$ makes it the Lerch-Mobius series. So ζ_F and M_F are two pairings of one set-only product G_L against one arithmetic-only kernel: the digit set enters through the positions and never through the primes. Nothing in the proof uses $k = q$, and this is what is left of an Euler product when Theorem 3.1 has removed the other one.

Remark 3.6 (What the position product is for). Taking $a = \mu$ turns a design Mobius sum into an integral of G_L against the Mobius exponential sum $S(x,\theta) = \sum_{n \le x} \mu(n) e(n\theta)$, for which Baker and Harman [3] prove $\max_\theta |S(x,\theta)| \ll_\varepsilon x^{3/4+\varepsilon}$ under the generalized Riemann hypothesis. That is the route by which a bound on M_F is usually attempted, and it is not pursued here; Theorem 3.5 is quoted because it is the exact statement of what a design has in place of an Euler product, and because it makes plain that the digit set and the arithmetic occupy separate factors.

Fact 3.7 (The position identity, checked)**.** The identity of Theorem 3.5 holds to 1.95×10^{-16} and 2.04×10^{-16} at base 10 missing 9 with $L = 3$ and $L = 4$, and to 2.9×10^{-16} at base 3 on $\{0,1\}$ with $L = 3, 4, 5$, at $s = 3.3$ and $s = 2.7 + 1.9i$, by Gauss-Legendre quadrature on the q^L cells. Script of record: `lab/mrly-euler`, verb `position`.

3.3. The replacement identity.

Theorem 3.8 (The Dirichlet inverse of a design)**.** *Let $1 \in F$. Then $\mathbf{1}_{S_F}$ has a unique Dirichlet inverse ν_F, given by $\nu_F(1) = 1$ and*

$$\nu_F(n) \;=\; -\sum_{\substack{d \mid n,\ d>1 \\ d \in S_F}} \nu_F(n/d) \qquad (n > 1),$$

so that $\zeta_F(s) N_F(s) = 1$ as formal Dirichlet series and pointwise wherever both converge absolutely. The support of ν_F lies inside the multiplicative semigroup generated by S_F, and strictly inside it. At the full digit set $\nu_F = \mu$.

The existence and uniqueness of a Dirichlet inverse for any f with $f(1) \neq 0$ is entirely classical and the one-line induction is recalled in Section 4.3 only to fix the recursion the script runs. What the statement adds is the location of the support. At base 3 on $\{0,1\}$ the integers 9, 27 and 36 lie in the semigroup generated by S_F and have $\nu_F = 0$, while $16 = 4 \cdot 4$, $48 = 4 \cdot 12$ and $52 = 4 \cdot 13$ lie in that semigroup and outside S_F. So four sets are in play and no two of them coincide: the design S_F; the semigroup it generates; the Beurling integers built on the primes lying in S_F, treated in Section 3.7; and the support of ν_F.

Fact 3.9 (The inverse, computed)**.** On the full digit set $\nu_F = \mu$ term for term to $n = 131072$ at $q = 2$ and to $n = 177147$ at $q = 3$. Off it, the partial sums of $N_F(\sigma)$ meet $1/\zeta_F(\sigma)$ to 1.60×10^{-3} at $\sigma = 0.8008$ and to 1.96×10^{-4} at $\sigma = 0.9208$ at base 3 on $\{0,1\}$, and to 1.72×10^{-2} at $\sigma = 1.0816$ and 2.39×10^{-3} at $\sigma = 1.2016$ at base 10 missing 9. All four abscissas sit above

the $\operatorname{Re}\rho$ of Fact 3.13 and nothing is evaluated below it. Script of record: `lab/mrly-pairing`, verb `inverse`.

3.4. **The transport.**

Theorem 3.10 (The transport)**.** *Let $1 \in F$ and suppose ζ_F has a zero ρ with $\operatorname{Re}\rho > \alpha$. Then*

$$\sigma_c(N_F) \;\geq\; \operatorname{Re}\rho,$$

and consequently $\sum_{n\leq x}\nu_F(n) \neq O(x^{\operatorname{Re}\rho-\varepsilon})$ for every $\varepsilon > 0$.

Remark 3.11 (What the transport is and is not)*.* The proof is the identity theorem applied on the connected half-plane $\operatorname{Re} s > \max(\sigma_c(N_F), \alpha)$, which is free of the pole lattice, plus the standard implication from a partial-sum bound to an abscissa. It is elementary, and it is stated as a theorem for what it is fed rather than for its depth. It is also strictly one-sided: it bounds $\sigma_c(N_F)$ from below with no error term and no hypothesis beyond $1 \in F$, and the converse inequality $\sigma_c(N_F) \leq \sup\operatorname{Re}\rho$ is not proved here or anywhere the author knows. In particular "the rightmost zero is the abscissa" is not a statement of this paper.

3.5. **Zeros to the right of the abscissa.** For a Dirichlet series with non-negative coefficients, having a zero inside the half-plane of absolute convergence is possible in general: $1 + 2^{-s}$ has abscissa $-\infty$ and zeros at $(2m+1)\pi i/\log 2$. The content here is not the principle but the object. A design zeta is an infinite series with a genuine finite abscissa, and it has zeros to the right of it, which the Euler product forbids for ζ.

Fact 3.12 (The zero census)**.** The census counts zeros of the Lyndon cofactor $Z(s) = \zeta_F(s)(1 - kq^{-s})$, which is analytic on $\operatorname{Re} s > \alpha - 1$, on the single strip $\alpha - 0.92 < \operatorname{Re} s < \alpha + 3.02$, $0.02 < \operatorname{Im} s < 60$, split at $\operatorname{Re} s = \alpha$ exactly, so nothing is left unscanned against the pole line. Base 3 on $\{0, 1\}$ carries 3 zeros to the right of the abscissa and 20 to the left; base 10 missing 9 carries 13 right and 25 left; base 3 on $\{0, 2\}$ carries 3 right, in the same three boxes as $\{0, 1\}$. The base 2 full digit set is the control and each side names its object: zeros of ζ in $\alpha + 0.02 < \operatorname{Re} s < \alpha + 3.02$ count 0, computed and not quoted; zeros of ζ in $\alpha - 0.98 < \operatorname{Re} s < \alpha - 0.02$ count 13, the first thirteen below $\operatorname{Im} s = 60$; and the teeth of $1 - 2q^{-s}$, which lie exactly on $\operatorname{Re} s = \alpha$ and are not zeros of ζ, bring the one-strip cofactor count to $19 = 13 + 6$ with $6 = \lfloor 60 \log 2/2\pi \rfloor$. To the right of the abscissa no continuation is used: the positive series converges absolutely there and the ladder only rearranges it. The largest surviving phase step on any census contour is 0.9896 and the largest propagated error bound met at any census evaluation is 9.99×10^{-11}, both printed beside every count. The count is resolved and not certified: nothing here bounds ζ_F'/ζ_F on the contour, so a zero pair closer than the surviving spacing would stay invisible. Script of record: `lab/design-zeta`.

Fact 3.13 (Two certified boxes)**.** The argument principle returns winding number 1 on the rectangle $\operatorname{Re} s \in [0.72074, 0.72084]$, $\operatorname{Im} s \in$

$[28.60563, 28.60573]$ at base 3 on $\{0,1\}$, with contour minimum $|\zeta_F| = 8.298 \times 10^{-4}$ against a propagated evaluation bound of 6.284×10^{-30}; and winding number 1 on $\operatorname{Re} s \in [1.00150, 1.00168]$, $\operatorname{Im} s \in [2.73915, 2.73925]$ at base 10 missing 9, with contour minimum 6.865×10^{-4} against 2.798×10^{-23}. The control rectangle $\operatorname{Re} s \in [0.99900, 1.00050]$, $\operatorname{Im} s \in [2.73810, 2.74030]$ at base 10 returns winding 0. Each box therefore contains exactly one zero, and five digits of $\operatorname{Re} \rho$ are certified in each case. Both boxes lie strictly to the right of the respective abscissa, $\alpha = 0.6309297536$ and $\alpha = 0.9542425094$. Script of record: `lab/mrly-pairing`, verb `box`, on the engine of `lab/design-zeta`.

Corollary 3.14 (Anti-cancellation)**.** *At base* 3 *on* $\{0,1\}$, $\sum_{n \le x} \nu_F(n) \neq O(x^{0.72074-\varepsilon})$ *for every* $\varepsilon > 0$, *while* $A_F(x) \asymp x^{0.6309297}$. *At base* 10 *missing* 9, $\sum_{n \le x} \nu_F(n) \neq O(x^{1.00150-\varepsilon})$ *for every* $\varepsilon > 0$, *and the exponent exceeds* 1.

So on these two designs the limit superior of the design's own Mertens function is strictly larger than the design's own mass, and at base 10 missing 9 it is larger than the count of all integers below x. A conjecture of the form "the design's Mobius partial sums are $O(x^{\alpha/2+\varepsilon})$" is therefore false for ν_F on these two designs and can only be carried by μ restricted to S_F, that is by M_F; and Theorem 3.3 is exactly what keeps the zeros of ζ_F from reaching M_F.

Fact 3.15 (The census is far below the limit superior)**.** The two statements of Corollary 3.14 are about a limit superior and the finite census does not see them. Pointwise the running maximum of $\sum_{n \le x} \nu_F(n)$ reaches only 0.0738 of $A_F(x)$ at base 3 on $\{0,1\}$ at level $L = 16$, and 0.0847 of x at base 10 missing 9 at $L = 7$. Its growth per level at base 10 missing 9 reads 9.4474, 11.5000, 10.2220, 10.0354 at $L = 4, \dots, 7$, against $q^{\operatorname{Re}\rho} = 10.036661$ and the trivial rate $k = 9$; only the last of the four lands on the predicted rate, so quoting it alone would be selection. The ratio of running maximum to $A_F(q^L)$ rises 0.1043, 0.1094, 0.1398, 0.1588, 0.1771 at $L = 3, \dots, 7$. At base 3 on $\{0,1\}$ the geometric mean of the four steps $L = 12, \dots, 16$ is 2.059 against $q^{\operatorname{Re}\rho} = 2.207512$ and the trivial $k = 2$, while the arithmetic mean of the five printed level ratios is 1.9972, below the trivial rate: at this depth the census cannot separate the two rates. Script of record: `lab/mrly-pairing`, verb `inverse`.

3.6. **The glue that gains nothing.** Since $\zeta_F M_F \neq 1$, write $\zeta_F M_F = 1 + D_F$ and ask whether D_F is cheap enough that $M_F = (1 + D_F)/\zeta_F$ buys anything. It is not.

Proposition 3.16 (The abscissa of the defect)**.** *Let* $0, 1 \in F$ *and* $k < q$. *Then* $\sigma_a(D_F) \le \alpha$. *Moreover, if* $\sigma_c(D_F) < \alpha$ *then* $M_F(\sigma) \to 0$ *as* $\sigma \to \alpha^+$ *along the real axis.*

The second half is Landau: ζ_F has nonnegative coefficients and abscissa α, so it is singular there and $\zeta_F(\sigma) \to +\infty$, while D_F would be holomorphic

at α. There is no circularity, since M_F is dominated termwise by ζ_F and so converges absolutely at every $\sigma > \alpha$ with no hypothesis at all.

Fact 3.17 (The defect is not cheap)**.** Write $P(x) = \sum_{n \le x} c_F(n)$ for the partial sums of the coefficients of $\zeta_F M_F$. If $\sigma_c(D_F)$ were below α then $P(x) = o(x^\alpha)$ would follow. Sampled at four phases per level, because $P(x)/x^\alpha$ is log-periodic and sampling only at $x = q^L$ aliases every Fourier mode onto one number, $P(x)/x^\alpha$ reads 0.493767, 0.699235, 0.758519, 0.587055 at base 3 on $\{0, 1\}$ out to $x = 3^{17.75}$, bounded away from 0 and from infinity. So $\sigma_c(D_F) = \alpha$ there. At base 10 missing 9 the same four phases read -0.011742, 0.033309, 0.058519, 0.073298 out to $x = 10^{6.75}$, which is a smaller and less settled reading and is not claimed as a determination. Script of record: `lab/mrly-pairing`, verb `glue`.

Corollary 3.18 (The glue is lossy)**.** *On a design where Theorem 3.10 applies, $M_F = (1 + D_F)N_F$ with $\sigma_c(N_F) > \alpha$, so factoring M_F through the design's own Mobius function replaces the trivial exponent α by something strictly worse. The route is not neutral, it is lossy.*

3.7. **Two other products, and what each one sees.** Two more multiplicative structures sit near a design, and each is worth stating precisely, because each is blind to something.

Theorem 3.19 (The Lyndon word product)**.** *Give the free monoid on the alphabet F the norm $N(w) = q^{|w|}$. Then*

$$\sum_w N(w)^{-s} \;=\; \frac{1}{1 - kq^{-s}} \;=\; \prod_{L \ge 1} \left(1 - q^{-Ls}\right)^{-c_k(L)},$$

where $c_k(L) = \frac{1}{L} \sum_{d|L} \mu(d) k^{L/d}$ is the number of Lyndon words of length L over a k-letter alphabet. This is an exact Euler product whose primes are the Lyndon words. This zeta has no zero, its own Mobius function is supported on the empty word and the single letters so its Mertens function is $1 - k$ at every norm above 1, and its poles are exactly the design pole lattice $s = \alpha + 2\pi i m / \log q$.

The reading is that the design's multiplicative shadow carries the pole lattice and no Riemann-hypothesis content whatever, and that all such content of ζ_F sits in the cofactor $Z(s) = \zeta_F(s)(1 - kq^{-s})$, which is precisely the function Fact 3.12 censuses.

Fact 3.20 (The Lyndon expansion)**.** The product of Theorem 3.19 expands to $1/(1 - ku)$ through u^{16} at $k = 2, 3, 4, 9, 10$, and $c_2(L)$ for $L = 1, \ldots, 10$ is $2, 1, 2, 3, 6, 9, 18, 30, 56, 99$, the sequence A001037 [1]. Script of record: `lab/mrly-euler`, verb `word`.

Theorem 3.21 (The Beurling system collapses at a non-unit digit gcd)**.** *Let N_F^B be the free abelian monoid on the primes lying in S_F, so that μ restricts to it and the Beurling zeta is $\prod_{p \in S_F} (1 - p^{-s})^{-1}$. If $\gcd(F) = a > 1$ then*

every element of S_F is a multiple of a; when a is prime the only prime of the design is a, N_F^B is the powers of a and the Beurling Mertens function $M_B(x)$ vanishes for $x \geq a$; when a is composite S_F contains no prime at all, $N_F^B = \{1\}$ and $M_B \equiv 1$.

So the Beurling route is blind to every scaled column, base 10 on $\{0, 4, 8\}$ for instance, while the scaling theorem of Section 3.8 reads those columns exactly. And N_F^B is not S_F: the two sets are different and the census below is a census on the wrong one, deliberately, as the control it is.

Fact 3.22 (The Beurling census to 10^6)**.** Base 3 on $\{0, 2\}$ has the single prime 2 and $M_B \equiv 0$ past 2. Base 3 on $\{0, 1\}$ has 525 primes, $N_F^B(920483) = 2198$, running maximum $|M_B| = 98$ and exponent $\log(\max)/\log x = 0.3339$ against $\alpha/2 = 0.3155$. Base 10 missing 9 has 35139 primes, $N_F^B(10^6) = 488864$ against $x^\alpha = 531441$, $M_B(10^6) = 1860$ with running maximum 1866, and exponent 0.4203, 0.4882, 0.5452 at $x = 10^4, 10^5, 10^6$ against $\alpha/2 = 0.4771$, where the full base 10 control reads 0.4084, 0.4241, 0.4276 at the same points against its own $\alpha/2 = 0.5$ and reproduces $M(10^4) = -23$, $M(10^5) = -48$, $M(10^6) = 212$, A084237 [2]. What this reads is a level and not a trend: $+0.068$ over $\alpha/2$ for the design against -0.072 for the control, with a running maximum climbing in both. There is cancellation, 0.545 against the trivial $\alpha = 0.954$, and it sits above $\alpha/2$. Script of record: `lab/mrly-euler`, verb `beurling`.

A Beurling system need not satisfy a Riemann hypothesis at all. Beurling introduced generalised prime systems in [4], and Diamond, Montgomery and Vorhauer [7] construct one whose integer counting function is $\kappa x + O(x^\theta)$ with $1/2 < \theta < 1$ and whose zeta still has infinitely many zeros on $\sigma = 1 - a/\log t$. So no regularity of the counting function of a design forces anything about M_B, and Fact 3.22 is offered as a measurement and not as evidence for a conjecture.

3.8. **Scaled digit sets share a zero set.**

Theorem 3.23 (Scaling)**.** *Let a be a positive integer with $a \max F \leq q-1$, so that $aF \subseteq \{0, \dots, q-1\}$. Then $m \mapsto am$ is a carry-free bijection $S_F \to S_{aF}$ and*

$$\zeta_{aF}(s) \;=\; a^{-s}\,\zeta_F(s).$$

The factor a^{-s} has no zero and no pole, so ζ_{aF} and ζ_F have exactly the same zeros, and residues in the ratio $a^{-s_{m,j}}$. The proof uses $0 \in F$ nowhere.

Fact 3.24 (Scaling, checked)**.** Base 3 on $\{0, 2\}$ against base 3 on $\{0, 1\}$: the identity of Theorem 3.23 holds to 5.6×10^{-43} at three points, and on the censused strip $\alpha < \operatorname{Re} s < \alpha + 3.02$, $0.02 < \operatorname{Im} s < 60$ the two censuses coincide box for box, winding 1 in $\operatorname{Im} s \in [22.01, 24.01]$, in $[28.01, 30.01]$ and in $[56.00, 58.00]$ for both and winding 0 in every other box. To the left of the abscissa $\{0, 2\}$ is not censused and is inferred from the theorem. Script of record: `lab/design-zeta`.

3.9. **Open problems.** The transport is one-sided and the paper leaves it there. Nothing above bounds $\sigma_c(N_F)$ from above, so it is not known whether the abscissa of the design's own Mobius series equals the supremum of the real parts of the zeros of ζ_F, or even whether that supremum is finite. Nothing above determines the locus of those zeros: the three at base 3 on $\{0,1\}$ below height 60 are located and counted, and no law is offered for where the next one sits, nor for how their number grows with height. Two of the four designs the generators carry, base 4 and base 5 on $\{0,1\}$, have no zero census at all, so Corollary 3.14 is silent on them and it is not known whether every proper design has a zero to the right of its abscissa or only some do. The zero counts of Fact 3.12 are resolved and not certified, and certifying them needs a bound on ζ_F'/ζ_F along a contour that no one here has. The determination $\sigma_c(D_F) = \alpha$ of Fact 3.17 rests on a measurement at one design and wants a proof. And the conjecture the whole apparatus was built to test, that $\sum_{n\le x, n\in S_F}\mu(n)$ is $O(x^{\alpha/2+\varepsilon})$, is untouched by everything above: Corollary 3.14 rules out ν_F as its carrier and says nothing whatever about μ restricted to S_F.

4. Proofs

4.1. **The wall.**

Proof of Theorem 3.1. Sufficiency is immediate: on the full digit set $\mathbf{1}_{S_F}$ is identically 1 on the positive integers, which is completely multiplicative.

For necessity, let F be proper. A multiplicative f has $f(1) = 1$, so $1 \in F$ is forced, and if $1 \notin F$ we are done. Assume $1 \in F$. The missing digits are nonempty, and either some missing digit is at least 2, or the only missing digit is 0.

Case A: some $c \in \{2,\dots,q-1\}$ is missing. Take the least such c, so $\{1,\dots,c-1\} \subseteq F$ and $c \notin F$. Write $R_a = (q^a-1)/(q-1)$ for the repunit with a ones. Its digits are all 1, so $R_c, R_{c+1} \in S_F$, and $\gcd(R_a,R_b) = R_{\gcd(a,b)}$ gives $\gcd(R_c,R_{c+1}) = R_1 = 1$. Now

$$R_cR_{c+1} \;=\; \sum_{i<c}\sum_{j<c+1} q^{i+j},$$

and the number of pairs (i,j) with $0 \le i \le c-1$, $0 \le j \le c$ and $i+j = m$ is $\min(m+1,\, c,\, 2c-m)$, which is at most $c \le q-1$. No carry occurs, so that count is literally the q^m digit of the product, and as m runs over $0,\dots,2c-1$ the digits run over exactly $\{1,\dots,c\}$. Since $c \notin F$ the product leaves S_F, while both factors lie in it, so $f(R_cR_{c+1}) = 0 \ne 1 = f(R_c)f(R_{c+1})$.

Case B: only 0 is missing, so $F = \{1,\dots,q-1\}$. If q is odd then $q \ge 3$ and $(q^2+1)/2 = \frac{q-1}{2}\,q + \frac{q+1}{2}$, whose two digits $\frac{q-1}{2}$ and $\frac{q+1}{2}$ both lie in $\{1,\dots,q-1\}$; moreover $q^2 \equiv 1 \pmod 8$, so $(q^2+1)/2$ is odd and is coprime to 2. Both 2 and $(q^2+1)/2$ lie in S_F, while the product q^2+1 has base-q digits $1,0,1$ and does not. If q is even then q^2-1 and q^2+1 are both odd and differ by 2, hence coprime; the product q^4-1 has every digit equal to

$q-1$ and so lies in S_F, while q^2+1 has the digit 0 and does not, so here $f(mn) = 1 \neq 0 = f(m)f(n)$. This case covers $q = 2$, $F = \{1\}$, with the pair $(3, 5)$ and product $15 = 1111_2$. □

Proof of Theorem 3.3. Write $c_F(n) = \sum_{de=n,\ d,e\in S_F} \mu(e)$ for the n-th coefficient of $\zeta_F M_F$. If $1 \notin F$ then $n = 1$ has no admissible factorisation and $c_F(1) = 0$, which is (i). If some prime $p \in S_F$ has $p^2 \notin S_F$, the divisors of p^2 are $1, p, p^2$, and the only factorisation with both sides in S_F is (p, p), so $c_F(p^2) = \mu(p) = -1$, which is (ii).

Suppose neither holds and $c_F(n) = 0$ for every $n > 1$. Let g be the least element of S_F greater than 1, so that no divisor of anything lying strictly between 1 and g is in S_F. Taking $n = g$, the admissible factorisations are $(1, g)$ and $(g, 1)$, so $0 = c_F(g) = \mu(g) + 1$ and $\mu(g) = -1$; in particular g is squarefree. Now induct: suppose $g^i \in S_F$ for all $i < j$, which holds vacuously at $j = 1$. In $c_F(g^j)$ write the factorisation as $d \cdot e$ with $e \in S_F$ and $\mu(e) \neq 0$, so e is a squarefree divisor of g^j, hence a divisor of g. If $e = 1$ then $d = g^j$ must lie in S_F and the term is $\mu(1) = 1$. If $e > 1$ then $e \in S_F$ forces $e \geq g$ and $e \mid g$ forces $e = g$, so $d = g^{j-1}$, which lies in S_F by the induction hypothesis, and the term is $\mu(g) = -1$. Hence $0 = c_F(g^j) = \mathbf{1}[g^j \in S_F] - 1$, so $g^j \in S_F$. That is (iii). At the full digit set $\zeta_F = \zeta$ and $M_F = 1/\zeta$, so the two are inverse. □

4.2. **The position product.**

Proof of Theorem 3.5. Expanding the product over positions,

$$G_L(t) \;=\; \sum_{(d_0,\dots,d_{L-1})\in F^L} e\!\left(\Big(\sum_{i<L} d_i q^i\Big)t\right),$$

so $\int_0^1 G_L(t)e(-nt)\,dt$ counts the strings in F^L with $\sum_{i<L} d_i q^i = n$. Since $F \subseteq \{0, \dots, q-1\}$, that is the number of base-q representations of n with L digits, all in F, which is 1 if $n \in D_L$ and 0 otherwise, by uniqueness of the expansion. For the second identity, $|G_L| \leq k^L$ is bounded and $\sum_n |a(n)|n^{-\operatorname{Re} s}$ converges, so the sum defining $A(s, t)$ converges uniformly in t and may be integrated term by term against G_L; the first identity then selects exactly the $n \in D_L$.

□

4.3. **The Dirichlet inverse and the transport.**

Proof of Theorem 3.8. Since $1 \in F$ we have $\mathbf{1}_{S_F}(1) = 1$, and the classical construction applies: define $\nu_F(1) = 1$ and, for $n > 1$, let $\nu_F(n)$ be given by the stated recursion, which is exactly the requirement $\sum_{d|n,\ d\in S_F} \nu_F(n/d) = 0$ solved for the $d = 1$ term. Uniqueness is the same recursion read forwards. The identity $\zeta_F N_F = 1$ is the resulting convolution $\mathbf{1}_{S_F} * \nu_F = \delta$ read as Dirichlet series, valid formally and, where both series converge absolutely, pointwise.

For the support, induct on n: $\nu_F(n)$ is a sum of terms $\nu_F(n/d)$ with $d \in S_F$, $d > 1$, so if $\nu_F(n) \neq 0$ then $\nu_F(n/d) \neq 0$ for some such d, and by induction n/d lies in the multiplicative semigroup generated by S_F; hence so does n. That the containment is strict is witnessed at base 3 on $\{0, 1\}$ by $9 = 3 \cdot 3$, 27 and 36, which lie in the semigroup and have $\nu_F = 0$. On the full digit set the recursion is $\nu_F(n) = -\sum_{d|n, d>1} \nu_F(n/d)$, which is the defining recursion of μ, so $\nu_F = \mu$. □

Proof of Theorem 3.10. Write $\sigma_0 = \sigma_c(N_F)$ and suppose, for contradiction, that $\sigma_0 < \operatorname{Re}\rho$. Put

$$U \;=\; \{s : \operatorname{Re} s > \max(\sigma_0, \alpha)\},$$

a connected half-plane. On U the series N_F converges and defines a holomorphic function, and ζ_F is holomorphic there as well, since its abscissa of absolute convergence is α and every pole of its continuation has real part at most α. On the sub-half-plane where both series converge absolutely, the convolution identity $\mathbf{1}_{S_F} * \nu_F = \delta$ gives $\zeta_F(s)N_F(s) = 1$. Both sides are holomorphic on the connected open set U, so by the identity theorem $\zeta_F N_F = 1$ throughout U, and in particular ζ_F has no zero in U. But $\operatorname{Re}\rho > \alpha$ and $\operatorname{Re}\rho > \sigma_0$ place $\rho \in U$, a contradiction. Hence $\sigma_0 \geq \operatorname{Re}\rho$.

For the last clause, if $\sum_{n \leq x} \nu_F(n) = O(x^\theta)$ then the standard partial-summation estimate gives $\sigma_c(N_F) \leq \theta$. Taking $\theta = \operatorname{Re}\rho - \varepsilon$ would contradict the bound just proved. □

Proof of Proposition 3.16. The n-th coefficient of $\zeta_F M_F$ is $c_F(n) = \sum_{de=n,\ d,e \in S_F} \mu(e)$, so $|c_F(n)|$ is at most the n-th coefficient of ζ_F^2, a Dirichlet series with nonnegative coefficients and abscissa of absolute convergence α. Hence $\sigma_a(D_F) \leq \alpha$.

For the second half, ζ_F has nonnegative coefficients and abscissa α, so by Landau's theorem it is singular at $s = \alpha$ and $\zeta_F(\sigma) \to +\infty$ monotonically as $\sigma \to \alpha^+$ along the real axis. If $\sigma_c(D_F) < \alpha$ then D_F is holomorphic at α and in particular bounded near it, so

$$M_F(\sigma) \;=\; \frac{1 + D_F(\sigma)}{\zeta_F(\sigma)} \;\longrightarrow\; 0.$$

There is no circularity: M_F is dominated termwise by ζ_F, hence converges absolutely at every $\sigma > \alpha$ with no hypothesis on the size of $\sum_{n \leq x, n \in S_F} \mu(n)$. □

4.4. **The two other products, and scaling.**

Proof of Theorem 3.19. The free monoid on F has k^L words of length L, each of norm q^L, so with $u = q^{-s}$ the norm zeta is $\sum_{L \geq 0} k^L u^L = 1/(1 - ku)$ for $\operatorname{Re} s > \alpha$. By the Chen-Fox-Lyndon theorem every word over F factors uniquely as a non-increasing product of Lyndon words, so the free monoid is

equinumerous by norm with the free abelian monoid on the Lyndon words; counting the latter by norm gives

$$\frac{1}{1-ku} \;=\; \prod_{L\ge 1}\left(1-u^{L}\right)^{-c_k(L)},$$

with $c_k(L)$ the number of Lyndon words of length L over a k-letter alphabet, which is $\frac{1}{L}\sum_{d\mid L}\mu(d)k^{L/d}$ by the standard necklace count. This is an Euler product with the Lyndon words as primes. It has no zero, since $1/(1-ku)$ never vanishes. Its own Mobius function is the coefficient sequence of $1-ku$, that is 1 at the empty word and -1 at each of the k letters and 0 beyond, so its summatory function is $1-k$ at every norm above 1. Its poles are the solutions of $kq^{-s}=1$, that is $s=\log_q k+2\pi i m/\log q=\alpha+2\pi i m/\log q$, which is the design pole lattice of Definition 2.2.

Since the free monoid is not the free abelian monoid on Lyndon words but only equinumerous with it by norm, this is a statement about norms and carries no arithmetic beyond them; and since $Z(s)=\zeta_F(s)(1-kq^{-s})$ divides out exactly this factor, whatever zero structure ζ_F has beyond the lattice sits in Z. □

Proof of Theorem 3.21. If $\gcd(F)=a>1$ then every base-q digit of every $n\in S_F$ is divisible by a, hence so is n. If a is prime, an element of S_F that is prime must therefore equal a, and $a\in S_F$ exactly when a is a one-digit element, which it is; so the primes of the design are $\{a\}$, N_F^B is $\{a^j : j\ge 0\}$, and its Mobius sum telescopes to $\mu(1)+\mu(a)=0$ at every $x\ge a$. If a is composite, no element of S_F is prime at all, $N_F^B=\{1\}$ and $M_B\equiv 1$. □

Proof of Theorem 3.23. Let $m\in S_F$ have base-q digits $d_i\in F$. Since $ad_i\le a\max F\le q-1$, the integer $am=\sum_i(ad_i)q^i$ is presented with digits ad_i, all lying in aF and all below q, so this is its base-q expansion and $am\in S_{aF}$. Conversely, if $n\in S_{aF}$ has digits $e_i\in aF$, then each $e_i=ad_i$ with $d_i\in F$, and $m=\sum_i d_iq^i$ lies in S_F with $am=n$. So $m\mapsto am$ is a bijection $S_F\to S_{aF}$ and

$$\zeta_{aF}(s) \;=\; \sum_{m\in S_F}(am)^{-s} \;=\; a^{-s}\zeta_F(s)$$

on $\operatorname{Re} s>\alpha$, hence on the whole plane by continuation. The factor a^{-s} is entire and never zero, so ζ_{aF} and ζ_F have the same zeros with the same multiplicities, and residues in the ratio $a^{-s_{m,j}}$. Nothing in the argument used $0\in F$. □

5. Reproducibility

Two scripts accompany this paper. Both are plain `python3` with no dependency beyond the standard library, take no arguments, write nothing outside `figures/`, and are run from the lane root. Every assertion names the value obtained and the value wanted, and a failure stops the script.

`python3 scripts/verify.py` runs in about one second in a few tens of megabytes and covers the following, in order.

- *The wall*, Theorem 3.1: all 246 nonempty digit sets of the bases $2 \le q \le 7$. Of these 120 have $1 \notin F$ and 6 are full; at each of the remaining 120 the constructed pair is built, asserted coprime, and asserted to break multiplicativity in one direction or the other. The branch split is 114 repunit, 3 odd base, 3 even base, and the pairs at base 3 on $\{0,1\}$, base 2 on $\{1\}$ and base 12 on $\{1\}$ are checked individually. This is a smaller domain than Fact 3.2, which the script does not rerun.
- *The inverse*, Theorem 3.8: ν_F at base 3 on $\{0,1\}$ computed to $n = 3^8 = 6561$ from the recursion, then the convolution $\mathbf{1}_{S_F} * \nu_F$ recomputed by an independent loop over $a \in S_F$ and asserted equal to δ at every $n \le 6561$, so that $\zeta_F N_F = 1$ is checked as a coefficient identity by a second code path. The semigroup generated by S_F is built to the same bound: 9, 27 and 36 are asserted inside it with $\nu_F = 0$, and 16, 48 and 52 inside it and outside S_F; and $\nu_F(n) \ne 0$ is asserted to imply membership at every $n \le 6561$. The control is the full digit set, where $\nu_F = \mu$ is asserted term for term to $n = 8192$ at $q = 2$ and $n = 6561$ at $q = 3$, against an independent sieve for μ.
- *The position product*, Theorem 3.5: the identity in its exact grid form. Since G_L is a trigonometric polynomial of degree below q^L, the integral equals the q^L-point Riemann sum exactly, so the check is an identity between finite sums and not a quadrature. Base 3 on $\{0,1\}$ at $L = 5$ for every $n < 243$, worst error 1.4×10^{-15}; base 10 missing 9 at $L = 3$ for every $n < 1000$, worst error 4.4×10^{-15}.
- *The Lyndon product*, Theorem 3.19: the expansion of $\prod_L (1 - u^L)^{-c_k(L)}$ through u^{16} in exact integer arithmetic, asserted equal to $\sum_i k^i u^i$ at $k = 2, 3, 4, 9, 10$, and $c_2(L)$ asserted against the first ten terms of A001037.
- *The design zeta and the two box centres*: the peel recursion of Definition 2.2, namely $(1 - kq^{-s})\zeta_F(s) = \sum_{a \in F, a \ne 0} a^{-s} + \sum_{l \ge 1} \binom{-s}{l} q^{-s-l} \gamma_l \zeta_F(s + l)$ with $\gamma_l = \sum_{a \in F} a^l$, which is Burnol's Proposition 4.1 [5], solved bottom-up over 260 integer shifts in double precision, the highest shifts evaluated by direct summation. It is validated at $s = 2$ against the direct sum over the elements of S_F below q^L, inside the proved truncation tail $k(kq^{-s})^L/(1 - kq^{-s})$: the discrepancy is 1.392×10^{-10} against a tail of 4.1×10^{-10} at base 3 on $\{0,1\}$ with $L = 15$, and 6.042×10^{-6} against 5.8×10^{-5} at base 10 missing 9 with $L = 5$. It is then re-evaluated at the two box centres of Fact 3.13, returning $|\zeta_F| = 7.46 \times 10^{-5}$ at base 3 and 1.22×10^{-5} at base 10, against 9.78×10^{-1} and 6.87×10^{-1} one unit higher in $\operatorname{Im} s$.

- *Scaling*, Theorem 3.23: $S_{2F} = 2S_F$ as sets below 3^9 at base 3, and $\zeta_{\{0,2\}}(s) = 2^{-s}\zeta_{\{0,1\}}(s)$ to 1.4×10^{-16} at $s = 0.6 + 3.1i$ through the same evaluator.

One caution about the last item, and it is the important one. The smallness of $|\zeta_F|$ at the two box centres is an *illustration and not a certificate*. It is exactly what a box five digits wide and a derivative of size $O(1)$ predict, and a small value of a function bounds nothing about its zeros without a lower bound on the derivative, which is taken nowhere here. What certifies each zero is the winding number of Fact 3.13, computed by the argument principle in `lab/mrly-pairing`, verb `box`, on the interval-bounded engine of `lab/design-zeta`. Note also that a raw truncation of the Dirichlet series cannot serve even as an illustration: to the right of the abscissa the tail after L digit levels is of size $(kq^{-\operatorname{Re}\rho})^L$, which is 0.906^L at base 3 on $\{0,1\}$ and 0.897^L at base 10 missing 9, so no reachable depth makes it small. The peel recursion is a resummation of the same series and is what makes the evaluation possible at all.

The script does not rerun the ten Facts of Section 3, namely ?? 3.2?? 3.4?? 3.7?? 3.9, ?? 3.12?? 3.13?? 3.15, and ?? 3.17?? 3.22?? 3.24. Those come from three generators of the public research tree at `https://github.com/mrlyprod/mrlyprod`, under `research/lab/`:

- `mrly-euler`, verbs `wall`, `pair`, `position`, `word`, `beurling`;
- `mrly-pairing`, verbs `inverse`, `glue`, `box`;
- `design-zeta`, the census and the evaluation engine.

Of these, `position` runs in a few minutes and every other `mrly-euler` verb in seconds; `glue` and `inverse` take about thirty seconds each and `box` about twenty; the full census of `design-zeta` to $\operatorname{Im} s = 60$ is the longest run in the group. Their exact finite domains are the ones stated in each Fact above and repeated in each generator's own documentation.

`python3 scripts/figure.py` runs in under a second. It writes `figures/figure.svg`, the picture the repository README embeds, and `figures/zeros.tex`, the Ti*k*Z version of Fig. 1 that this document includes; both are drawn from the same data, namely the exact pole lattice $\alpha + 2\pi ij/\log q$, the certified box of Fact 3.13, the two uncertified height bands of Fact 3.12, and the first six ordinates of ζ on the control column. `tectonic paper.tex` rebuilds `paper.pdf`.

Acknowledgments

This paper was developed and verified in collaboration with Claude (Anthropic). The author takes sole responsibility for every claim.

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