DIVISOR AVATARS: WHICH PARITY DESIGNS COUNT THE DIVISORS OF A POWER

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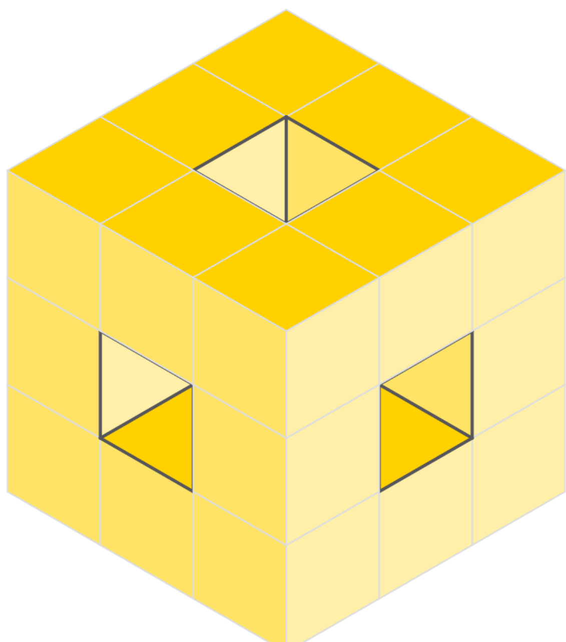

Abstract. Colour the cells of a cube by whether their coordinates are odd or even, keep a chosen set of colours, and count what you kept. This paper shows that the resulting count is often the number of divisors of a power, and settles exactly when. For an integer x with $\omega(x) = D$ distinct prime factors, a design of dimension D whose cell count at side $2n+1$ equals $d(x^n)$ for every $n \geq 0$ exists if and only if $\Omega(x) \leq 2\omega(x)$; when it exists its weight signature is forced, and the number of such designs is a product of binomial coefficients. The qualifying divisor polynomials in dimension D number $\sum_{m\leq D} p(m)$, the partition sums A000070. The Menger sponge is the boundary case: its fill is $d(240^n)$ and its void count is A395241.

1. Introduction

Take a cube of $3 \times 3 \times 3$ small cells and number the coordinates $1, 2, 3$. Throw away every cell that has at least two even coordinates. Seven cells go: the one in the very middle, and the six in the middles of the faces. Twenty stay. That is the first step of the Menger sponge, and it is drawn in Fig. 1.

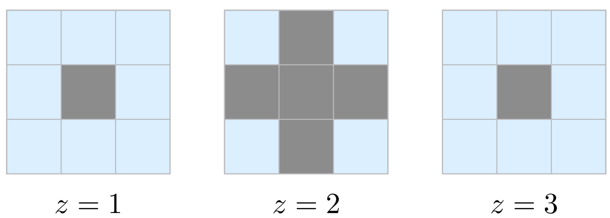

Figure 1. The first Menger step at side 3, one layer at a time. Dark cells have at least two even coordinates and are removed; 7 go and 20 stay. Now $20 = d(240)$, and the coincidence is not one: at side $2n+1$ the same rule keeps exactly $d(240^n)$ cells.

Now enlarge the cube to side 5, then 7, then $2n+1$, applying the same rule. The counts are $1, 20, 81, 208, 425, \ldots$. They are the divisor counts of the powers of 240:

$$d(240^n) = (4n+1)(n+1)^2, \qquad d(240^0) = 1,\ d(240) = 20,\ d(240^2) = 81.$$

So a purely geometric census – how many cells survive an odd/even rule – is computing an arithmetic function. That is the phenomenon this paper is about.

The rule is completely specified by which parity patterns it keeps. In dimension D a *design* is a set F of parity patterns, a subset of $\{0,1\}^D$, and the design's *fill* at side $2n+1$ is the number of cells whose parity pattern lies in F. The Menger rule is the design that keeps every pattern with at most one even coordinate. Call a design a *divisor avatar* of x when its fill equals $d(x^n)$ at every $n \geq 0$.

Two questions follow immediately. Which integers x have an avatar? And how many designs realise one? The answer to the first is a single clean inequality between the two standard prime-counting functions: $\Omega(x)$, the number of prime factors with multiplicity, and $\omega(x)$, the number of distinct ones.

> **An integer $x > 1$ has a divisor avatar if and only if $\Omega(x) \leq 2\,\omega(x)$.**

The Menger sponge sits exactly on the boundary: $240 = 2^4 \cdot 3 \cdot 5$ has $\Omega = 6$ and $\omega = 3$. Push one prime factor further – to 480, with $\Omega = 7$ – and no design in any dimension has that fill. The obstruction is not subtle once you see it: the fill of a D-dimensional design cannot count more than $\binom{D}{1} = D$ parity patterns of weight one, and the arithmetic demands $\sum_i (a_i - 1)$ of them. When the inequality does hold the design is essentially forced: its weight signature is determined outright, and only the choice of which patterns to take at each weight is free, giving $\prod_w \binom{\binom{D}{w}}{f_w}$ designs. Counting the qualifying polynomials in dimension D turns into counting partitions, and the answer is $\sum_{m \leq D} p(m)$, the sequence A000070 [6]: $2, 4, 7, 12, 19, 30, \ldots$

What is and is not new here. The fill of a design is an Ehrhart-type count: lattice points of a dilated box that lie in prescribed cosets of $(2\mathbb{Z})^D$, and such counts are quasi-polynomial by the standard theory [4]. Restricted to odd sides they are honest polynomials, and Lemma 3.1 is a two-line special case, not an application of the general machinery. The formula $d(x^n) = \prod_i (a_i n + 1)$ is classical [3]. The change of basis that turns one into the other – substituting $a_i n + 1 = (a_i - 1)n + (n + 1)$ – is one line and no depth is claimed for it.

What is new is the pair of questions above and their answers: the realizability criterion $\Omega(x) \leq 2\omega(x)$ (Theorem 3.5), the count $\prod_w \binom{\binom{D}{w}}{f_w}$ of designs realizing it, the A000070 census (Corollary 3.9), and the splitting test that decides an avatar from its fill polynomial alone (Corollary 3.8). The closed form for the Menger void count is the author's own OEIS entry A395241 [8], whose name field already records $a(n) = n^2(4n+3)$ and whose comments already record the Menger reading and the anchor $a(1) = 7$; Proposition 3.10 supplies that entry's missing derivation and is presented as such, not as a new result.

Section 2 sets up designs and fills. Section 3 states everything. Section 4 proves it. A short arithmetic tour rides along: which divisor-sum functions can and cannot be counted by a polynomial (Proposition 3.13), what happens to Robin's inequality along the seven three-dimensional avatar ladders (Proposition 3.14), and a machine scan of every three-dimensional design against nine observables (Fact 3.21).

2. Definitions

Throughout, $D \geq 0$ is an integer, $p_1 < p_2 < \cdots$ are the primes, d is the number-of-divisors function, $\sigma_k(m) = \sum_{e|m} e^k$ and $\sigma = \sigma_1$, $\omega(x)$ is the number of distinct primes dividing x and $\Omega(x)$ the number counted with multiplicity. For a vector $b = (b_1, \ldots, b_D)$, $e_w(b)$ is the w-th elementary symmetric polynomial, with $e_0(b) = 1$ and $e_w(b) = 0$ for $w > D$. Finally $p(m)$ is the number of partitions of m.

Definition 2.1 (design)**.** The *parity cube* in dimension D is $\{0,1\}^D$; its elements are called *corners*, and the *weight* of a corner c is $|c| = c_1 + \cdots + c_D$. A *design* is a subset $F \subseteq \{0,1\}^D$. Its *weight signature* is the vector $f = (f_0, \ldots, f_D)$ with

$$f_w = \#\{\, c \in F : |c| = w \,\}.$$

In dimension $D = 0$ the parity cube $\{0,1\}^0$ has a single element, the empty corner, of weight 0; so dimension zero has exactly two designs, $\emptyset$ and $\{()\}$, of signatures (0) and (1).

Definition 2.2 (fill)**.** Let $B_n = \{1, 2, \ldots, 2n+1\}^D$ be the cubical grid of odd side $2n+1$, its cells addressed by integer coordinates. The *parity* of a cell $u \in B_n$ is the corner $\pi(u) \in \{0,1\}^D$ with $\pi(u)_i = 1$ when u_i is even and $\pi(u)_i = 0$ when u_i is odd. A cell is *filled* by F when $\pi(u) \in F$, and the *fill* of F is

$$P_F(n) = \#\{\, u \in B_n : \pi(u) \in F \,\}, \qquad n \geq 0.$$

The *void* of F is $V_F(n) = (2n+1)^D - P_F(n)$.

Definition 2.3 (divisor avatar)**.** A design F of dimension D is a *divisor avatar* of the integer $x > 1$ when

$$P_F(n) = d(x^n) \quad \text{for every integer } n \geq 0.$$

We then call x an *avatar integer* of F, and the least such x the *minimal avatar*.

Definition 2.4 (Menger design)**.** The *Menger design* is $F_M = \{\, c \in \{0,1\}^3 : |c| \leq 1 \,\}$, of signature $(1,3,0,0)$. Its filled cells are exactly those with at most one even coordinate, so its voids are exactly the cells removed by the first step of the Menger sponge construction at side $2n+1$.

Definition 2.5 (the touched complex)**.** Realise each cell $u \in B_n$ as the closed unit box $\prod_i [u_i - 1, u_i] \subset \mathbb{R}^D$. A vertex, edge or 2-face of the unit grid of $\mathbb{R}^D$ is *touched* by F when it lies in the box of some cell filled by F; write $\mathrm{ver}_F(n)$, $\mathrm{edg}_F(n)$, $\mathrm{fac}_F(n)$ for the numbers of touched vertices, edges and 2-faces. Two filled cells are *adjacent* when their boxes share a $(D-1)$-face; let $\mathrm{adj}_F(n)$ count the adjacent pairs, $\kappa_F(n)$ the connected components of the filled cells under adjacency, and $\mathrm{sur}_F(n)$ the $(D-1)$-faces lying in exactly one filled box. The *cycle rank* of F is $\mathrm{adj}_F(n) - P_F(n) + \kappa_F(n)$, and in dimension three the *Euler characteristic* of F is

$$\chi_F(n) = \mathrm{ver}_F(n) - \mathrm{edg}_F(n) + \mathrm{fac}_F(n) - P_F(n).$$

3. Results

The fill is a polynomial, and the polynomial remembers the design.

Lemma 3.1 (fill law)**.** *For every design F of dimension D and every $n \geq 0$,*

$$P_F(n) = \sum_{w=0}^{D} f_w \,(n+1)^{D-w}\, n^w.$$

In particular P_F agrees on $n \geq 0$ with a polynomial of degree D and leading coefficient $|F|$ when $F \neq \emptyset$, and with the zero polynomial otherwise.

Lemma 3.2 (the fill basis is a basis)**.** *The $D+1$ polynomials $q_w(n) = (n+1)^{D-w}n^w$, $w = 0, \dots, D$, are linearly independent over $\mathbb{Q}$. Consequently the fill polynomial of a design determines its weight signature.*

Lemma 3.3 (realizability)**.** *A vector $f = (f_0, \dots, f_D)$ of integers is the weight signature of some design of dimension D if and only if $0 \leq f_w \leq \binom{D}{w}$ for every w. The number of designs with that signature is $\prod_{w=0}^{D} \binom{\binom{D}{w}}{f_w}$.*

Lemma 3.4 (binomial domination)**.** *Let $b_1, \dots, b_D$ be nonnegative integers with $m = b_1 + \cdots + b_D$. Then*

$$0 \leq e_w(b) \leq \binom{m}{w} \qquad (0 \leq w \leq D),$$

and hence $e_w(b) \leq \binom{D}{w}$ for every w whenever $m \leq D$.

The criterion.

Theorem 3.5 (avatar criterion)**.** *Let $x > 1$ be an integer with $x = q_1^{a_1} \cdots q_D^{a_D}$ over its $D = \omega(x)$ distinct prime divisors $q_1 < \cdots < q_D$, every $a_i \geq 1$, and put $b_i = a_i - 1$. Then:*

(i) *$d(x^n) = \sum_{w=0}^{D} e_w(b)\,(n+1)^{D-w} n^w$ for every $n \geq 0$, as an identity of polynomials in n;*

(ii) *a design F of dimension D satisfies $P_F(n) = d(x^n)$ for every $n \geq 0$ if and only if its weight signature is $f_w = e_w(b)$ for every w;*

(iii) *such a design exists if and only if*

$$\Omega(x) \;\leq\; 2\,\omega(x), \qquad \textit{equivalently} \qquad \sum_{i=1}^{D} (a_i - 1) \leq D;$$

(iv) *when it exists, the number of such designs is exactly $\displaystyle\prod_{w=0}^{D} \binom{\binom{D}{w}}{e_w(b)}$;*

(v) *no design of any dimension other than D has fill polynomial $d(x^n)$.*

Remark 3.6. Part (iv) is the reason one must say "any design with signature $e_w(b)$" and never "the design". For $x = 240$ the signature $(1, 3, 0, 0)$ in dimension 3 is realised by $\binom{1}{1}\binom{3}{3}\binom{3}{0}\binom{1}{0} = 1$ design, so the Menger design is unique;

but for $x = 2 \cdot 3 \cdot 5$ the signature is $(1, 0, 0, 0)$, realised by $\binom{1}{1}\binom{3}{0}\binom{3}{0}\binom{1}{0} = 1$ design, while for $x = 2^2 \cdot 3 \cdot 5$ the signature $(1, 1, 0, 0)$ is realised by $\binom{3}{1} = 3$ distinct designs, all with the same fill.

Remark 3.7. The failure is real and not rare. The smallest failing x is $x = 8$: $D = 1$, $b = (2)$, and the demanded signature $(1, 2)$ needs $f_1 = 2 > \binom{1}{1} = 1$. In dimension two, $x = 72 = 2^3 3^2$ demands $(1, 3, 2)$ against the caps $(1, 2, 1)$; exhaustive search over all 16 designs of dimension two confirms that $d(72^n) = 6n^2 + 5n + 1$ is not among the 12 distinct fill polynomials available there.

Corollary 3.8 (splitting criterion). *Let F be a nonempty design of dimension D. Then F is a divisor avatar of some integer x with $\omega(x) = D$ if and only if*

$$P_F(0) = 1 \quad \text{and} \quad P_F \text{ splits into linear factors over } \mathbb{Q}.$$

In that case $P_F(n) = \prod_{i=1}^{D}(a_i n + 1)$ with every a_i a positive integer, the avatar integers of F are exactly the $\prod_i p_{\tau(i)}^{a_i}$ over injections τ into the primes, and the minimal avatar is obtained by pairing the largest slope with the smallest prime.

Corollary 3.9 (census of qualifying polynomials). *For every $D \geq 0$ the number of distinct polynomials of the form $d(x^n)$ with $\omega(x) = D$ that arise as the fill of some design of dimension D equals*

$$\sum_{m=0}^{D} p(m) = \mathrm{A000070}(D).$$

For $D = 0, 1, 2, \ldots, 8$ these are $1, 2, 4, 7, 12, 19, 30, 45, 67$.

The Menger sponge, both halves.

Proposition 3.10 (void census). *In the cubical grid of side $2n + 1$, the number of cells having at least two even coordinates is*

$$a(n) = 3n^2(n + 1) + n^3 = n^2(4n + 3),$$

with $a(1) = 7$: the central cell and the six face centres. This is the closed form recorded in A395241 [8]; the proposition supplies its derivation.

Corollary 3.11 (the sponge is a boundary avatar). *The Menger design F_M of Definition 2.4 has*

$$P_{F_M}(n) = (n + 1)^2(4n + 1) = 4n^3 + 9n^2 + 6n + 1 = d(240^n),$$
$$V_{F_M}(n) = (2n + 1)^3 - P_{F_M}(n) = n^2(4n + 3),$$

so its fill is a divisor avatar with minimal avatar 240 and its void is A395241. Moreover $\Omega(240) = 6 = 2\,\omega(240)$, so 240 lies exactly on the boundary of Theorem 3.5(iii), while 480, with $\Omega = 7$ and $\omega = 3$, has no avatar in any dimension.

Remark 3.12. Corollary 3.11 is the complement identity $P + V = (2n + 1)^3$ with both halves already on record: the fill polynomial follows from Lemma 3.1 by expansion, and the void is A395241 [8]. It is offered as the paper's worked example, not as a discovery.

Which divisor functions a polynomial can count.

Proposition 3.13 (divisor sums are excluded)**.** *Let Q be a polynomial in n with rational coefficients. Then:*

(i) *$Q(n) = \sigma_k(x^n)$ for every $n \geq 0$ is impossible for every integer $k \geq 1$ and every $x > 1$;*
(ii) *$Q(n) = \sigma(x^n)/x^n$ for every $n \geq 0$ is impossible for every $x > 1$.*

At $x = 1$ both expressions collapse to the constant $1 = d(1^n)$. Since every fill is a polynomial (Lemma 3.1), no design in any dimension has fill $\sigma_k(x^n)$ or $\sigma(x^n)/x^n$ for any $x > 1$: among the functions σ_k only $k = 0$, the divisor count itself, is ever the fill of a design.

The seven three-dimensional ladders and Robin's inequality. Exactly seven integers occur as minimal avatars of three-dimensional designs, namely $30, 60, 120, 180, 240, 360, 900$; they are 5-smooth because a minimal avatar in dimension three puts its exponents on the primes $2, 3, 5$, which is forced by minimality and is not an arithmetic accident. Robin's theorem [5] states that the Riemann Hypothesis is equivalent to $\sigma(N) < e^{\gamma} N \log\log N$ for all $N > 5040$.

Proposition 3.14 (the corridors recede, for all n)**.** *Let $x > 1$ have prime support $\{2, 3, 5\}$, which covers all seven ladders. Then for every $n \geq 1$*

$$1 < \frac{\sigma(x^n)}{x^n} < \frac{15}{4}, \qquad \lim_{n\to\infty} \frac{\sigma(x^n)}{x^n} = \frac{15}{4}, \qquad \lim_{n\to\infty} \frac{\sigma(x^n)}{x^n \log\log(x^n)} = 0.$$

Consequently $\sigma(N) < e^{\gamma} N \log\log N$ holds for every such power $N = x^n$ with

$$\log\log N > \frac{15}{4e^{\gamma}} = 2.10547306\ldots, \qquad \textit{that is} \qquad N > 3681.171813\ldots,$$

and in particular for every one of them exceeding 5040*. No computation is needed for any of this.*

Fact 3.15 (the corridor table)**.** Over the exact finite domain consisting of the 140 powers $N = x^n$ with $x \in \{30, 60, 120, 180, 240, 360, 900\}$ and $1 \leq n \leq 20$, exactly 131 of the powers satisfy $N > 5040$, and every one of those 131 satisfies Robin's inequality. The domain is 140 powers but only 130 distinct integers, since $900^n = 30^{2n}$; the 131 powers above 5040 are 122 distinct integers. The largest ratio $\sigma(N)/(N \log\log N)$ among them occurs at $N = 14400 = 120^2$ and equals 1.573259905933 to twelve decimal places, against $e^{\gamma} = 1.781072417990$. Checked by `scripts/verify.py` with exact integer σ and 40-digit decimal arithmetic.

Remark 3.16. Fact 3.15 is illustration, not evidence: Proposition 3.14 already proves the inequality on all seven ladders for all n, with no computation. And neither bears on the Riemann Hypothesis. Robin's criterion quantifies over all $N > 5040$; it suffices to check the colossally abundant numbers [5], and the least counterexample, should one exist, must be superabundant [1]. Verifying the inequality on a sparse family of powers excludes nothing. Identities describe; only a bound over the full domain constrains.

Colossally abundant numbers.

Fact 3.17 (the first colossally abundant number with no avatar)**.** Over the exact finite domain consisting of the first 13 colossally abundant numbers A004490 [7], namely

$$2,\ 6,\ 12,\ 60,\ 120,\ 360,\ 2520,\ 5040,$$
$$55440,\ 720720,\ 1441440,\ 4324320,\ 21621600,$$

the first twelve satisfy $\Omega \leq 2\omega$ and therefore have divisor avatars by Theorem 3.5, while the thirteenth,

$$21621600 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13, \qquad \Omega = 13, \quad \omega = 6,$$

has $\Omega = 13 > 12 = 2\omega$ and therefore has no avatar: no design, in any dimension, has fill $d(21621600^n)$. Checked by `scripts/verify.py`.

Remark 3.18. Fact 3.17 corrects a natural guess. Colossally abundant numbers have exactly the shape $p_1^{a_1} p_2^{a_2} \cdots p_D^{a_D}$ with $a_1 \geq \cdots \geq a_D \geq 1$ [2], which is the minimal-avatar shape, and this makes it tempting to claim that every colossally abundant number is a minimal avatar in dimension ω. That claim is false, and 21621600 refutes it. The correct statement is the criterion: a colossally abundant number has an avatar exactly when $\Omega \leq 2\omega$. Surjectivity fails too, and badly: dimension D carries A000070(D) qualifying polynomials against a handful of colossally abundant numbers.

A machine scan in dimension three. The group generated by coordinate permutations and coordinate reversals has order 48 and acts on the 256 designs of dimension three with 22 orbits. Coding a design as $\sum_{c \in F} 2^{c_1 + 2c_2 + 4c_3}$, take the least code in each orbit as its representative. Note that a coordinate reversal moves the odd/even origin, so it does *not* preserve the fill at a fixed odd side; the scan below therefore classifies the 22 canonical least-mask representatives, not all 256 oriented masks.

Before the scan, one general fact about the observables of Definition 2.5. The fill is a disjoint union of boxes, which is why Lemma 3.1 is a one-line count. The touched counts are unions of boxes that overlap, so they need inclusion-exclusion; the point of the next lemma is that the overlaps stay inside a small family of intervals whose sizes are linear in n.

Lemma 3.19 (box unions are polynomial)**.** *For $n \geq 0$ set*

$$O = \{1, 3, \ldots, 2n+1\}, \qquad E = \{2, 4, \ldots, 2n\},$$
$$O^+ = \{0, 1, \ldots, 2n+1\}, \qquad E^+ = \{1, 2, \ldots, 2n\},$$

and let $\mathcal{S}_n = \{\emptyset,\ O \cap E^+,\ O,\ E,\ E^+,\ O^+\}$, of cardinalities $0, n, n+1, n, 2n, 2n+2$. Then $\mathcal{S}_n$ is closed under intersection, and if $B_1, \ldots, B_r$ are boxes $B_k = \prod_{i=1}^{D} U_{k,i}$ with every $U_{k,i} \in \mathcal{S}_n$ chosen by a rule independent of n, then $|B_1 \cup \cdots \cup B_r|$ agrees for all $n \geq 0$ with a polynomial in n of degree at most D.

Proposition 3.20 (the local counts are polynomial)**.** *Let F be a design of dimension D. Then each of ver_F, edg_F, fac_F, sur_F and adj_F agrees for all $n \geq 0$ with a polynomial in n of degree at most D. Consequently, in dimension three the Euler characteristic χ_F agrees for all $n \geq 0$ with a polynomial of degree at most three; and in any dimension the cycle rank agrees with a polynomial on a given set of sides exactly when the component count κ_F does, since the two differ by $\mathrm{adj}_F - P_F$.*

Fact 3.21 (extended avatar scan)**.** Over the exact finite domain consisting of the 22 least-mask orbit representatives of dimension three, the nine observables

fill, voids, surface, touched vertices, touched edges,
touched faces, Euler characteristic, components, cycle rank,

and the eleven sides $2n+1$ for $n = 0, 1, \ldots, 10$ (largest grid $21^3 = 9261$ cells, every count literal, no sampling), the following hold. Every one of the 198 observable sequences agrees on that range with a polynomial of degree at most three except the component and cycle-rank sequences of codes 30 and 126, which do so only for $n \geq 1$. Among the sequences that are of the form $d(x^n)$ for all n in range with $x > 1$, exactly eight are not fills:

code	design	observable	law
0	$\emptyset$	voids	$(2n+1)^3 = d(900^n)$
1	$\{000\}$	Euler characteristic	$(n+1)^3 = d(30^n)$
1	$\{000\}$	components	$(n+1)^3 = d(30^n)$
3	$\{000, 100\}$	Euler characteristic	$(n+1)^2 = d(6^n)$
3	$\{000, 100\}$	components	$(n+1)^2 = d(6^n)$
7	$\{000, 100, 010\}$	components	$n+1 = d(2^n)$
15	$\{000, 100, 010, 110\}$	Euler characteristic	$n+1 = d(2^n)$
15	$\{000, 100, 010, 110\}$	components	$n+1 = d(2^n)$

There are in addition exactly nine sequences that are constantly $1 = d(1^n)$: components for the codes $23, 27, 31, 61, 63, 111, 127, 255$ and Euler characteristic for code 255. Finally, exactly eight representatives have a fill of the form $d(x^n)$ with $x > 1$, namely codes $1, 3, 7, 15, 23, 27, 63, 255$ with minimal avatars $30, 60, 120, 180, 240, 180, 360, 900$. Checked by `scripts/verify.py`, which recomputes all nine observables cell by cell, recovers each law by exact rational interpolation, and applies Corollary 3.8.

Conjecture 3.22 (the component count is polynomial in every dimension)**.** *For every design F in every dimension D there is a polynomial that agrees with the component count $\kappa_F(n)$ for all sufficiently large n.* Evidence: *Fact 3.21 exhibits such a polynomial for all* 22 *orbit representatives in dimension three over $n = 0, \dots, 10$, of degree at most three in every case, agreeing from $n = 0$ for twenty of them and from $n = 1$ for the two exceptions, codes* 30 *and* 126*. By Proposition 3.20 the same statement for the cycle rank is equivalent to this one; the corresponding statements for the touched counts and for the Euler characteristic are not conjectural but proved, and hold from $n = 0$.* Failure modes: *the component count is the one observable here that is not a union-of-boxes census, so Lemma 3.19 does not reach it; two filled regions can merge into one component only at sides above a threshold, and nothing in the present argument bounds that threshold. The count could be quasi-polynomial with a nontrivial period, or eventually polynomial with more exceptional small sides than the two seen in dimension three, and nothing here bounds the number of exceptional sides in dimension $D \geq 4$.*

4. Proofs

Proof of Lemma 3.1. Among $1, 2, \dots, 2n+1$ there are exactly $n+1$ odd values and exactly n even values. Fix a corner $c \in \{0,1\}^D$. A cell $x \in B_n$ has $\pi(x) = c$ precisely when, for each coordinate i independently, x_i is even if $c_i = 1$ and odd if $c_i = 0$. The choices in distinct coordinates are independent, so the number of cells with $\pi(x) = c$ is $n^{|c|}(n+1)^{D-|c|}$. Summing over $c \in F$ and grouping the corners of equal weight gives the stated formula. Each term $q_w(n) = (n+1)^{D-w}n^w$ is a monic polynomial of degree D, so P_F is a polynomial of degree at most D whose coefficient of n^D is $\sum_w f_w = |F|$; this is nonzero exactly when $F \neq \emptyset$. □

Proof of Lemma 3.2. Suppose $\sum_{w=0}^{D} \lambda_w q_w = 0$ with $\lambda_w \in \mathbb{Q}$. We show $\lambda_0 = \cdots = \lambda_D = 0$ by induction on w. Evaluating at $n = 0$ kills every term with $w \geq 1$ and leaves $\lambda_0(0+1)^D = \lambda_0$, so $\lambda_0 = 0$. Assume $\lambda_0 = \cdots = \lambda_{j-1} = 0$ for some $j \geq 1$. Then $\sum_{w \geq j} \lambda_w (n+1)^{D-w} n^w = 0$ identically, and every term is divisible by n^j in $\mathbb{Q}[n]$; dividing by n^j gives $\sum_{w \geq j} \lambda_w (n+1)^{D-w} n^{w-j} = 0$ identically. Evaluating at $n = 0$ now kills every term with $w > j$ and leaves $\lambda_j = 0$. This completes the induction. Since $P_F = \sum_w f_w q_w$ by Lemma 3.1, and a polynomial identity holding for all $n \geq 0$ holds identically, the coefficients f_w are determined by P_F. □

Proof of Lemma 3.3. There are exactly $\binom{D}{w}$ corners of weight w, so any signature must satisfy $0 \leq f_w \leq \binom{D}{w}$. Conversely, given such a vector, choose any f_w of the weight-w corners for each w and take F to be the union;

the choices at distinct weights are independent, so this both realises f and counts the designs realising it as $\prod_w \binom{\binom{D}{w}}{f_w}$. $\square$

Proof of Lemma 3.4. For $A = \sum_w A_w t^w$ and $B = \sum_w B_w t^w$, say that A is dominated by B, written $A \preceq B$, when $0 \le A_w \le B_w$ for every w. Domination is preserved by products of polynomials with nonnegative coefficients: if $A \preceq A'$ and $B \preceq B'$ then the coefficient of t^w in AB is $\sum_j A_j B_{w-j} \le \sum_j A'_j B'_{w-j}$, the coefficient of t^w in $A'B'$.

For a nonnegative integer b we have $1 + bt \preceq (1+t)^b$, since the coefficients on the left are $1, b, 0, 0, \ldots$ and those on the right are $1, b, \binom{b}{2}, \ldots$, all nonnegative. Multiplying these D dominations together,

$$\sum_w e_w(b)\, t^w \;=\; \prod_{i=1}^{D}(1+b_i t) \;\preceq\; \prod_{i=1}^{D}(1+t)^{b_i} \;=\; (1+t)^m,$$

which is exactly $0 \le e_w(b) \le \binom{m}{w}$ for every w. If moreover $m \le D$ then $\binom{m}{w} \le \binom{D}{w}$ for every $w \ge 0$, since $\binom{m}{w}$ is nondecreasing in m. $\square$

Proof of Theorem 3.5. (i) Since $x^n = \prod_i q_i^{a_i n}$ with the q_i distinct, $d(x^n) = \prod_{i=1}^{D}(a_i n + 1)$. Write $b_i = a_i - 1 \ge 0$. Then

$$a_i n + 1 = (a_i - 1)n + (n+1) = b_i n + (n+1),$$

so, expanding the product by choosing from each factor either the term $b_i n$ or the term $(n+1)$,

$$\begin{aligned}\prod_{i=1}^{D}(a_i n + 1) &= \prod_{i=1}^{D}(b_i n + (n+1)) \\ &= \sum_{S \subseteq \{1,\ldots,D\}} \Big(\prod_{i \in S} b_i\Big) n^{|S|}(n+1)^{D-|S|} \\ &= \sum_{w=0}^{D} e_w(b)\, n^w (n+1)^{D-w},\end{aligned}$$

the last step grouping the subsets by their size. This is an identity in $\mathbb{Z}[n]$. (It is the generating-function identity $\prod_i (b_i t + 1) = \sum_w e_w(b) t^w$ evaluated at $t = n/(n+1)$ and multiplied by $(n+1)^D$; the expansion above avoids the division and is valid at $n = -1$ as well.)

(ii) By Lemma 3.1, $P_F = \sum_w f_w q_w$; by (i), $d(x^n) = \sum_w e_w(b) q_w(n)$. Two polynomials agreeing at all integers $n \ge 0$ are equal, so $P_F(n) = d(x^n)$ for all $n \ge 0$ is equivalent to $\sum_w (f_w - e_w(b)) q_w = 0$, and by Lemma 3.2 that forces $f_w = e_w(b)$ for every w. The converse is immediate from the same two displays.

(iii) By (ii) and Lemma 3.3, a design with fill $d(x^n)$ exists if and only if $0 \le e_w(b) \le \binom{D}{w}$ for every w. Necessity: taking $w = 1$ gives $e_1(b) = \sum_i (a_i - 1) = \Omega(x) - \omega(x) \le \binom{D}{1} = D = \omega(x)$, that is $\Omega(x) \le 2\omega(x)$.

Sufficiency: if $\sum_i b_i = m \le D$ then Lemma 3.4 gives $0 \le e_w(b) \le \binom{D}{w}$ for every w, so the signature is realizable.

(iv) Immediate from (ii) and the counting half of Lemma 3.3.

(v) A design F' of dimension $D' \ge 0$ has fill either identically zero (when $F' = \emptyset$) or a polynomial of degree exactly D', by Lemma 3.1. The polynomial $d(x^n) = \prod_{i=1}^{D}(a_i n + 1)$ has degree exactly D and is not identically zero, so $D' = D$. □

Proof of Corollary 3.8. Suppose first that F is a divisor avatar of x with $\omega(x) = D$. Then $P_F(n) = \prod_i (a_i n + 1)$ with each $a_i \ge 1$ an integer, which both splits over $\mathbb{Q}$ and gives $P_F(0) = 1$.

Conversely, suppose $P_F(0) = 1$ and P_F splits into linear factors over $\mathbb{Q}$. Since $F \neq \emptyset$, Lemma 3.1 gives $\deg P_F = D$ and $P_F \in \mathbb{Z}[n]$ with constant term $P_F(0) = f_0 = 1$. Write $P_F(n) = \sum_{j=0}^{D} c_j n^j$ with $c_0 = 1$, $c_D = |F| > 0$, and let

$$P_F^*(n) \;=\; n^D P_F(1/n) \;=\; \sum_{j=0}^{D} c_j n^{D-j} \;=\; n^D + c_1 n^{D-1} + \cdots + c_D,$$

a *monic* polynomial with integer coefficients. Because $P_F(0) = 1 \neq 0$, no root of P_F is zero, and the roots of P_F^* are exactly the reciprocals of the roots of P_F. By hypothesis all D roots of P_F are rational, hence so are all D roots of P_F^*; and by the rational root theorem every rational root of a monic integer polynomial is an integer. So the roots of P_F are $-1/a_1, \ldots, -1/a_D$ for nonzero integers $a_1, \ldots, a_D$ (the sign is a labelling choice).

Each a_i is positive. Indeed, by Lemma 3.1, $P_F(t) = \sum_w f_w (t+1)^{D-w} t^w$ has nonnegative coefficients f_w not all zero, so for real $t > 0$ every term is nonnegative and at least one is strictly positive, giving $P_F(t) > 0$; and $P_F(0) = 1 > 0$. Hence P_F has no root in $[0, \infty)$, so every root $-1/a_i$ is negative, so every a_i is positive.

Finally, $n + 1/a_i = (a_i n + 1)/a_i$, so $P_F(n) = \lambda \prod_i (a_i n + 1)$ for some rational λ, and setting $n = 0$ gives $\lambda = P_F(0) = 1$. Thus $P_F(n) = \prod_i (a_i n + 1)$, and for any injection τ from $\{1, \ldots, D\}$ into the primes, $x = \prod_i p_{\tau(i)}^{a_i}$ satisfies $\omega(x) = D$ and $d(x^n) = \prod_i (a_i n + 1) = P_F(n)$.

Every avatar integer has that shape. Indeed, let $y > 1$ be any avatar integer of F and write $y = r_1^{c_1} \cdots r_{D'}^{c_{D'}}$ over its distinct primes with every $c_i \ge 1$. Then $\prod_i (c_i n + 1) = d(y^n) = P_F(n) = \prod_i (a_i n + 1)$ as polynomials, and a polynomial determines its multiset of roots, so $D' = D$ and $\{c_1, \ldots, c_D\} = \{a_1, \ldots, a_D\}$ as multisets. Hence the avatar integers of F are exactly the $\prod_i p_{\tau(i)}^{a_i}$.

It remains to minimise over τ. If some τ uses a prime p_k while a smaller prime p_j, $j < k$, is unused, then replacing the factor p_k^a by p_j^a gives another avatar integer, with the same exponent multiset, that is strictly smaller because $a \ge 1$ and $p_j < p_k$. So a minimiser uses exactly the primes $p_1, \ldots, p_D$, and minimising x is minimising $\prod_i p_i^{a_{\rho(i)}}$ over orderings ρ, which by the rearrangement inequality applied to $\sum_i a_{\rho(i)} \log p_i$ is achieved by pairing the largest exponent with the smallest prime. $\square$

Proof of Corollary 3.9. By Theorem 3.5, the polynomials in question are exactly the $\prod_{i=1}^{D}(a_i n+1)$ for multisets $\{a_1, \dots, a_D\}$ of positive integers with $\sum_i (a_i - 1) \leq D$, and distinct multisets give distinct polynomials because a polynomial determines its multiset of roots. Substituting $b_i = a_i - 1$, these are exactly the multisets of D nonnegative integers with $\sum_i b_i \leq D$. Dropping the zero entries, such a multiset is a partition of some integer m with $0 \leq m \leq D$ into at most D positive parts; and since every part is at least 1, a partition of $m \leq D$ automatically has at most D parts. So the count is $\sum_{m=0}^{D} p(m)$, which is A000070(D) [6]. For $D = 0, \dots, 8$ the partial sums of $p(m) = 1, 1, 2, 3, 5, 7, 11, 15, 22$ are $1, 2, 4, 7, 12, 19, 30, 45, 67$. $\square$

Proof of Proposition 3.10. Number the coordinates from 1 to $2n + 1$; there are n even values and $n + 1$ odd values. A removed cell has either exactly two even coordinates or exactly three, and the two cases are disjoint. For exactly two: choose which two of the three coordinates are even, in $\binom{3}{2} = 3$ ways; each even coordinate then has n possible values and the remaining odd coordinate has $n + 1$, giving $3n^2(n + 1)$. For exactly three: n^3. Hence

$$a(n) = 3n^2(n + 1) + n^3 = 3n^3 + 3n^2 + n^3 = 4n^3 + 3n^2 = n^2(4n + 3).$$

At $n = 1$ this is 7, namely the central cell $(2, 2, 2)$ and the six face centres $(2, 2, 1), (2, 2, 3), (2, 1, 2), (2, 3, 2), (1, 2, 2), (3, 2, 2)$. $\square$

Proof of Corollary 3.11. The Menger design has signature $(1, 3, 0, 0)$, so the fill law of Lemma 3.1 gives

$$P_{F_M}(n) = (n+1)^3 + 3(n+1)^2 n = (n+1)^2\big((n+1) + 3n\big) = (n+1)^2(4n+1).$$

Since $240 = 2^4 \cdot 3 \cdot 5$ we have $d(240^n) = (4n + 1)(n + 1)(n + 1)$, which is the same polynomial; equivalently $b = (3, 0, 0)$ and $e_w(b) = (1, 3, 0, 0)$, matching Theorem 3.5(ii). Subtracting from $(2n + 1)^3 = 8n^3 + 12n^2 + 6n + 1$ gives $V_{F_M}(n) = 4n^3 + 3n^2 = n^2(4n + 3)$, which is Proposition 3.10 and A395241, as it must be since the cells F_M discards are exactly those with at least two even coordinates. Finally $\Omega(240) = 4+1+1 = 6$ and $\omega(240) = 3$, so $\Omega = 2\omega$; whereas $480 = 2^5 \cdot 3 \cdot 5$ has $\Omega = 7 > 6 = 2\omega$, so by Theorem 3.5(iii) and (v) no design in any dimension has fill $d(480^n)$. $\square$

Proof of Proposition 3.13. (i) For $k \geq 1$ and $x > 1$ we have $\sigma_k(x^n) \geq x^{kn} \geq 2^n$, since x^n is itself a divisor of x^n and $x^k \geq 2$. A polynomial in n of degree δ grows like n^δ, while $2^n/n^\delta \to \infty$. So the two cannot agree for all large n, hence not for all $n \geq 0$.

(ii) Write $x = \prod_{p^e \| x} p^e$. Then

$$\frac{\sigma(x^n)}{x^n} = \prod_{p^e \| x} \left(1 + p^{-1} + \cdots + p^{-en}\right).$$

This is a finite product of factors, each at least 1 and each strictly increasing in n (the factor for p gains the positive term $p^{-en-1} + \cdots + p^{-en-e}$ when n increases by one), so the product is strictly increasing in n. It is bounded above by $\prod_{p|x}(1-1/p)^{-1}$, a finite constant. A polynomial that is bounded on the nonnegative integers must be constant, since a nonconstant polynomial is unbounded there; but a strictly increasing sequence is not constant. Contradiction.

At $x = 1$, $\sigma_k(1) = 1$ and $\sigma(1)/1 = 1$ for every n, so both expressions are the constant $1 = d(1^n)$. $\square$

Proof of Proposition 3.14. Write $x = 2^\alpha 3^\beta 5^\gamma$ with $\alpha, \beta, \gamma \geq 1$; each of 30, $60, 120, 180, 240, 360, 900$ has this form. Then

$$\frac{\sigma(x^n)}{x^n} = \prod_{p\in\{2,3,5\}} (1 + p^{-1} + \cdots + p^{-e_p n}), \qquad (e_2, e_3, e_5) = (\alpha, \beta, \gamma).$$

Each factor is a truncation of a convergent geometric series, so it is strictly less than $(1-1/p)^{-1}$ and increases to it as $n \to \infty$. Hence the product is strictly less than $2 \cdot \frac{3}{2} \cdot \frac{5}{4} = \frac{15}{4}$ and tends to $\frac{15}{4}$; it exceeds 1 because every factor does. Since $\log\log(x^n) = \log(n \log x) \to \infty$, the third limit follows from the boundedness of the first ratio.

For the threshold: for $N = x^n$,

$$\frac{\sigma(N)}{N \log\log N} < \frac{15/4}{\log\log N},$$

and the right side is less than e^γ exactly when $\log\log N > \dfrac{15}{4e^\gamma}$. Numerically $e^\gamma = 1.781072417990\ldots$, so $15/(4e^\gamma) = 2.105473063376\ldots$ and $\exp\exp(15/(4e^\gamma)) = 3681.171813\ldots$. Every such power exceeding 3682 therefore satisfies Robin's inequality, and in particular every one exceeding 5040. $\square$

Proof of Lemma 3.19. Write $O^- = O \cap E^+ = \{1, 3, \ldots, 2n-1\}$. Among the six members of $\mathcal{S}_n$ one has the inclusions $O^- \subseteq O \subseteq O^+$, $O^- \subseteq E^+ \subseteq O^+$ and $E \subseteq E^+$, so every pair related by inclusion intersects inside the family. The pairs not so related are (O, E) and (O^-, E), which meet in $\emptyset$ because one member consists of odd and the other of even integers, and (O, E^+), which meets in O^- by definition. So $\mathcal{S}_n$ is closed under pairwise intersection, hence, being finite, under arbitrary intersections. Note that this table records which member the intersection is, and it is the same table for every $n \geq 0$.

Now let $B_k = \prod_i U_{k,i}$ be as stated. Intersections of boxes are computed slotwise, $\bigcap_{k\in G} B_k = \prod_i \big(\bigcap_{k\in G} U_{k,i}\big)$, so by the previous paragraph every such intersection is again a box all of whose slots carry members of $\mathcal{S}_n$, with the member in each slot named by a label that does not depend on n. Its cardinality is the product of the D slot cardinalities, each one of $0, n, n+1, n, 2n, 2n+2$, hence a polynomial in n of degree at most D. By inclusion-exclusion,

$$|B_1 \cup \cdots \cup B_r| = \sum_{\emptyset \neq G \subseteq \{1,\ldots,r\}} (-1)^{|G|+1} \Big| \bigcap_{k \in G} B_k \Big|,$$

a fixed integer combination, the same for every n, of polynomials of degree at most D. □

Proof of Proposition 3.20. Write $T_0 = O$, $T_1 = E$, $T_0^+ = O^+$, $T_1^+ = E^+$, all members of $\mathcal{S}_n$. As in the proof of Lemma 3.1, the cells of B_n of parity c are exactly the box $\prod_i T_{c_i}$, and these boxes are pairwise disjoint over distinct corners c.

Vertices. The vertices of the box of a cell u are the points v with $v_i \in \{u_i - 1, u_i\}$ for every i. For a single slot, $\bigcup_{u_i \in O}\{u_i - 1, u_i\} = O^+$ and $\bigcup_{u_i \in E}\{u_i - 1, u_i\} = E^+$, and conversely each $v_i \in T_{c_i}^+$ is realised by some $u_i \in T_{c_i}$: if v_i has the parity demanded by c_i take $u_i = v_i$, otherwise take $u_i = v_i + 1$, which stays in range in both cases. The slots are independent, so the vertices touched by the cells of parity c are exactly the box $\prod_i T_{c_i}^+$, and $\mathrm{ver}_F(n) = \big|\bigcup_{c \in F} \prod_i T_{c_i}^+\big|$. Apply Lemma 3.19.

Edges and 2-faces. An edge of the unit grid in direction j is addressed by a cell coordinate in slot j and a vertex coordinate in every other slot; the direction-j edges of the box of u are those with slot-j value u_j and slot-i value in $\{u_i - 1, u_i\}$ for $i \neq j$. By the same slotwise argument the direction-j edges touched by the cells of parity c form the box carrying T_{c_j} in slot j and $T_{c_i}^+$ elsewhere, so their union over $c \in F$ is covered by Lemma 3.19; summing over the D directions gives edg_F. A 2-face spanning slots j and k is addressed by cell coordinates in those two slots and vertex coordinates elsewhere, so the same argument with T_{c_j}, T_{c_k} in slots j, k and $T_{c_i}^+$ elsewhere, summed over the $\binom{D}{2}$ pairs, gives fac_F. Likewise a $(D-1)$-face omitting slot j is addressed by a vertex coordinate in slot j and cell coordinates elsewhere, giving $T_{c_j}^+$ in slot j and T_{c_i} elsewhere; call the resulting total $\Phi_F(n)$, again a polynomial of degree at most D. In dimension three $\Phi_F = \mathrm{fac}_F$, a 2-face and a $(D-1)$-face being the same object there.

Adjacent pairs and surface. Two cells are adjacent exactly when they differ by e_j in a single slot j, and $\pi(u + e_j)$ is $\pi(u)$ with slot j flipped. So the lower cells of the direction-j adjacent pairs of filled cells are those u with $\pi(u) = c$ for some $c \in F$ with $c \oplus e_j \in F$, and $u_j \leq 2n$: for each such c this is the box carrying $T_{c_j} \cap E^+$ in slot j and T_{c_i} elsewhere, with $O \cap E^+$ and $E \cap E^+ = E$ both in $\mathcal{S}_n$, and these boxes are disjoint over distinct c. Summing over j and applying Lemma 3.19 makes adj_F a polynomial of degree at most D. Finally, a $(D-1)$-face omitting slot j, at vertex coordinate v_j, lies in the boxes of exactly those filled cells u with the given coordinates elsewhere and $u_j \in \{v_j, v_j + 1\}$, so it lies in at most two filled boxes and in two precisely when that pair is adjacent. Hence $\Phi_F = \mathrm{sur}_F + \mathrm{adj}_F$ and $\mathrm{sur}_F = \Phi_F - \mathrm{adj}_F$ is a polynomial of degree at most D.

Euler characteristic and cycle rank. In dimension three $\chi_F = \mathrm{ver}_F - \mathrm{edg}_F + \mathrm{fac}_F - P_F$, and P_F is a polynomial of degree at most three by Lemma 3.1, so χ_F agrees with such a polynomial at every $n \geq 0$. In any dimension the cycle rank is $\mathrm{adj}_F - P_F + \kappa_F$, and $\mathrm{adj}_F - P_F$ is a polynomial, so on any set of sides the cycle rank agrees with a polynomial if and only if κ_F does. □

5. Reproducibility

One script, `scripts/verify.py`, checks every number in this paper. It is plain `python3` with no dependencies beyond the standard library, takes no arguments, reads no files, and is run from the lane root as `python3 scripts/verify.py`. Total runtime is under two seconds on a laptop. Every assertion names both the value obtained and the value wanted, and the script exits nonzero on the first disagreement; a clean run prints one line per domain point and the word `all green`.

- *Fill law* (Lemma 3.1). All 256 designs of dimension three, all sides $2n+1$ for $n = 0, \ldots, 6$: the cell-by-cell count is compared with $\sum_w f_w (n+1)^{3-w} n^w$.
- *Criterion* (Theorem 3.5). Every integer x with $2 \leq x \leq 3000$: the script factors x, tests $\Omega(x) \leq 2\omega(x)$, and independently searches all 2^{2^D} designs of dimension $D = \omega(x)$ for one whose cell-by-cell fill matches $d(x^n)$ at $n = 0, \ldots, 6$, for $D \leq 3$. The two verdicts must agree for every x in range, and where a design exists the count of designs found must equal $\prod_w \binom{\binom{D}{w}}{e_w(b)}$.
- *Counterexamples* (Remark 3.7). $x = 8$ in dimension one and $x = 72$ in dimension two: the script enumerates the 4 and 16 designs respectively and asserts that $d(x^n)$ is absent from the fill polynomials, reporting the 12 distinct fill polynomials of dimension two.
- *Census* (Corollary 3.9). Dimensions $D = 0, \ldots, 8$: the number of distinct qualifying polynomials is enumerated directly from multisets and asserted equal to $\sum_{m \leq D} p(m)$, giving $1, 2, 4, 7, 12, 19, 30, 45, 67$.
- *Sponge and void* (Proposition 3.10, Corollary 3.11). Sides $2n+1$ for $n = 0, \ldots, 10$: the removed cells are counted literally and compared with $n^2(4n+3)$, the kept cells with $d(240^n)$, and their sum with $(2n+1)^3$.
- *Robin corridors* (Fact 3.15). The 140 powers x^n with $x \in \{30, 60, 120, 180, 240, 360, 900\}$ and $1 \leq n \leq 20$: σ is computed exactly as an integer from the factorisation, the ratio in 40-digit decimal arithmetic. The script asserts that the 140 powers are 130 distinct integers, that 131 of the 140 exceed 5040 and are 122 distinct integers, that all 131 satisfy Robin's inequality, and that the maximum ratio is attained at $N = 14400$ with value 1.573259905933 to twelve places. It also checks the threshold constant $\exp\exp(15/(4e^\gamma)) = 3681.17\ldots < 5040$.

- *Colossally abundant* (Fact 3.17). The first 13 terms of A004490: the criterion $\Omega \leq 2\omega$ holds for the first twelve and fails at 21621600.
- *Extended scan* (Fact 3.21). The 22 least-mask orbit representatives of dimension three, nine observables, $n = 0, \ldots, 10$: every observable is recomputed cell by cell from the grid (components by union-find on face adjacency, cycle rank as graph edges minus fill plus components, Euler characteristic as touched vertices minus touched edges plus touched faces minus fill), each law is recovered by exact rational Lagrange interpolation and asserted to be a polynomial of degree at most three over the full range or over $n \geq 1$, and the splitting test of Corollary 3.8 is applied to every one of the 198 sequences, the four late-starting ones included, over the full range $n = 0, \ldots, 10$. The script asserts exactly the eight non-fill rows, exactly the nine constant identities, and exactly the eight fill avatars with their minimal avatars.

The figure in the lane README is regenerated by `scripts/figure.py`, also plain `python3`, run as `python3 scripts/figure.py`. It takes no input and writes `figures/sponge.svg` in well under a second.

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