BASE-3 DIGIT DESIGNS: DIAGONALITY, RAY MASSES, AND A SPECTRAL GAP AT TWO

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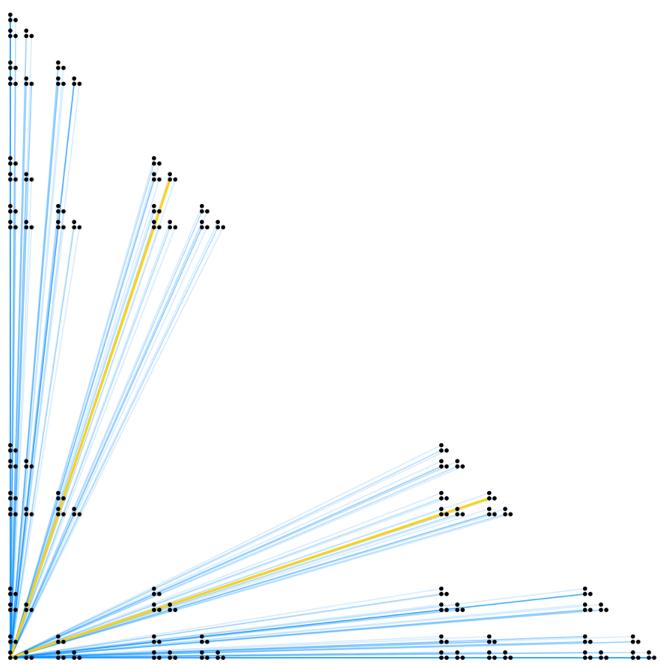
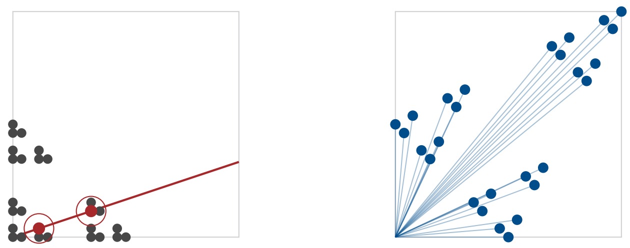


ABSTRACT. Draw a picture by writing numbers in base three and allowing only three of the nine possible digit pairs at each position; then ask which straight lines through the origin hit the picture, and how often. This paper answers three such questions for two families of base-three digit designs in the plane. First, a permutation design $F_\phi = \{(j, \phi(j))\}$ is *diagonal*, meaning no two distinct points of any level share a ray through the origin, exactly when $\phi(0) \neq 0$; this holds for four of the six permutations of $\{0, 1, 2\}$, at every level, and is proved here. Second, for the Sierpiński gasket design $G = \{(0, 0), (1, 0), (0, 1)\}$ the number of points on a fixed ray obeys an exact linear recurrence at every level: Fibonacci on $(3, 1)$, Narayana's cows on $(1, 12)$, and $c(m) = c(m-1) + c(m-4)$ on $(7, 3)$, each the characteristic polynomial of a live carry automaton with at most four states; on the shift ray $(3^j, 1)$ the mass is a product of j Fibonacci numbers, because the admissible multipliers are the binary strings with no two ones at distance j. That product bound is sharp enough to settle the whole shift family: the shift rays contribute $\big((13+5\sqrt{5})/11+o(1)\big)\varphi^{2n}$ to the second moment $\sum_{(a,b)} M_n(a, b)^2$, never more than $2.803\,\varphi^{2n}$, so the dominant carrier of that moment grows like $\varphi^{2n} = 2.618\ldots^n$ and not like 3^n. Third, for a coprime pair (s, t) the growth rate of $\{z : z, sz, tz \in G_n\}$ is the spectral radius of an explicit carry automaton, trimmed to the states that can return to where they started, and over all 829 coprime unordered pairs with $\max(s, t) \leq 52$ that radius equals 3 on the three shift pairs, equals 2 on exactly twenty pairs, and never lands in the open interval $(2, 3)$; for the variant automaton that drops the requirement that z itself lie in the gasket, the first three of those statements hold verbatim and the ceiling below 2 rises to 1.8488475886. Fourth, that variant automaton is a constrained tensor square of a single-coordinate carry automaton, and the collinear pair count is reorganised by the *witness* of a pair rather than by its multipliers: every witness has weight at least four, which pins the largest multiplier at level n to $\lfloor 3^n/8 \rfloor$; every multiplier pair above $(3^n - 1)/10$ carries exactly four collinear pairs; the weight layers scale exactly by three; and the weight-four layer is closed in Fibonacci, contributing less than $1.6945\,\varphi^{2n}$, the same golden law as the shift family. Fifth, the golden ceiling $M_n(z) \leq F(n+1) - 1$ on ray mass, which says the shift ray $(1, 3)$ is the heaviest ray of the gasket at every level, is proved rather than enumerated on a box of 13158 directions and at every level: a ray automaton has out-degree at most two with its branch states in one residue class modulo three, which forces a Fibonacci recursion whenever no branch state has two branching successors, and that settles all but twelve directions of the box, three shift rays closed by a Fibonacci product identity and nine by explicit rational certificates. Sixth, that whole case analysis collapses into one criterion: weighting the first returns of a ray automaton by φ^{-1} per step gives a single number $U(a, b)$ in $\mathbf{Q}(\sqrt{5})$, and $U(a, b) \leq \varphi^{-2}$ implies the ceiling for that direction at every level. It holds on 858 of the 865 occupied directions of the box and of six adversarial families, the seven exceptions being exactly the shift rays, where $U = \varphi^{-1}$ and the Fibonacci product argument applies instead. Seventh, that criterion is proved outright on an infinite arithmetic family: with $q = 3^k q_1$ the coordinate divisible by three and p the other, mass forces $q_1 \equiv p$ modulo three, and $k = 1$ with $v_3(q_1 - p) \leq 2$ gives $U \leq \varphi^{-2}$. A stdlib-only script that ships with the paper recomputes every number reported here, with one flagged exception: the second moment at levels 15, 16 and 17 runs the identical direction-grouping pass on a machine-integer array sort, because 3^{17} points do not fit the script's time budget.

1. Introduction



the gasket G_3: the ray $(3,1)$ holds 2 points   the design F_ϕ, $\phi = (1,0,2)$: 27 points, 27 rays

Figure 1. Two base-3 digit designs at level 3. On the left the gasket piles many points onto a few lines. On the right a diagonal design spreads every point onto a line of its own.

Write a nonnegative integer in base three. Now write two at once, stacked, so that position i carries a pair of digits (d_i^x, d_i^y) from the nine possibilities $\{0,1,2\}^2$. Choose three of those nine pairs and forbid the other six. The points you can still write form a self-similar cloud in the plane, and different choices of three give visibly different clouds. The question this paper asks is the simplest one available: which lines through the origin pass through the cloud, and how many of its points does each line catch? Two very different answers appear, depending on the design.

On the right of Fig. 1 is a *diagonal* design: no two of its 27 points share a line through the origin, so the cloud meets 27 distinct rays. This is not an accident of level three. Theorem 3.2 says that among the six designs of the form $F_\phi = \{(0,\phi(0)),(1,\phi(1)),(2,\phi(2))\}$, where ϕ permutes $\{0,1,2\}$, exactly four are diagonal at every level, and the test is as short as a test can be: $\phi(0) \neq 0$. The proof is elementary and complete. It rests on one observation, that every permutation of $\{0,1,2\}$ is an affine map $x \mapsto \alpha x + \beta$ over the field of three elements, and that after a cancellation the entire question collapses to whether $\beta = \phi(0)$ is zero.

On the left is the opposite extreme, the base-3 Sierpiński gasket $G = \{(0,0),(1,0),(0,1)\}$. Here the points crowd onto lines, and the counting becomes interesting. Call $M_n(a,b)$ the number of nonzero level-n gasket points on the ray through a primitive (a,b). Then $M_n(3,1)$ runs 0, 1, 2, 4, 7, 12, 20, 33, one less than a Fibonacci number each time, and $M_n(1,12)$ runs 0, 0, 1, 2, 3, 5, 8, 12, 18, one less than Narayana's cows [4]. These are exact identities at every level, not fits. The mechanism is not deep, and we say so plainly in Section 3: an admissible digit string on a fixed ray is a word in a small carry automaton, and for these rays the automaton is a run-length constraint, so the counting sequences are the classical composition counts. What is new is the dictionary, not the arithmetic behind it.

The shift rays $(3^j, 1)$ are the reason to care. Their masses are products of j Fibonacci numbers, and Theorem 3.9 turns that into a bound on the whole family at once: the second moment $\sum_{(a,b)} M_n(a,b)^2$ receives $((13 + 5\sqrt{5})/11 + o(1))\varphi^{2n}$ from the shift rays and never more than $2.803\,\varphi^{2n}$. Since $\varphi^2 = 2.618\ldots$ is below 3, the heaviest rays in the gasket are not heavy enough to make that moment grow like $3^n \cdot 3^n$, or even like $3^n \cdot \varphi^n$. Fact 3.14 measures what is left over.

The third question is about growth rather than counting. Fix coprime (s, t) and ask how many gasket points z have sz and tz also in the gasket. That set is again read by a carry automaton, and its growth rate is the spectral radius $\lambda(s, t)$ of the automaton once it is trimmed to the states that can get back to where they started (Proposition 3.16; the trimming is not optional, see Remark 2.8). The answer has a hole in it. Over all 829 coprime unordered pairs with $\max(s, t) \leq 52$, the radius is 3 on the three shift pairs $(1, 3)$, $(1, 9)$, $(1, 27)$, is exactly 2 on twenty further pairs, and otherwise sits at $1.6956\ldots$ or below. Nothing lands strictly between 2 and 3. The automaton is standard equipment; the hole is not.

The paper is short and self-contained. Section 2 fixes the objects. Section 3 states the three laws, with tags: what is proved is labelled Theorem, Proposition, or Lemma and proved in Section 4; what is checked over a stated finite range is labelled Fact, with that range written into the statement; what is believed is labelled Conjecture, with its evidence and the way it could fail. Section 5 names the script that recomputes every number.

Relation to existing work. The tool is textbook. A set defined by a digit restriction in base k is k-automatic, and reading it with a finite automaton whose states are carries is the standard method for such sets [1]; the growth exponent of the accepted language is the Perron root of the transition matrix [5]. The same device computes box dimensions of closed k-automatic sets, and base-3 multiplication automata are one of its stock examples [2]. Nothing in Section 3 claims the machine as new. What the paper contributes is arithmetic output: an exact classification of the diagonal permutation designs, an exact dictionary between three gasket rays and three classical composition counts, and the observation that these particular automata have no Perron root in $(2, 3)$, with an explicit largest value below 2 and the pairs that attain it.

The mass laws deserve the same honesty. Narayana's cows counts compositions of n into parts 1 and 3, and the Fibonacci numbers count binary strings with no two adjacent ones; the recurrence $c(m) = c(m-1) + c(m-4)$ counts compositions into parts 1 and 4. These are the most classical run-length counts there are. The content of Theorem 3.8 is that particular gasket rays realise them, not that the sequences are surprising; the same goes for the Fibonacci product on the shift rays, which is the no-two-ones constraint read along j interleaved subsequences. What is not routine is Theorem 3.9, where those products are summed against each other across all j at once and the total is pinned to a constant times φ^{2n}.

For the diagonality question we found no prior statement. The nearest literature is on radial projections and visible points of self-similar sets. A finite classification of the six base-3 permutation designs by a collinearity test is small enough that it is more likely unrecorded than known and forgotten; we state it, prove it, and make no priority claim.

2. Definitions

Throughout, $\mathbf{F}_3 = \{0,1,2\}$ with arithmetic modulo 3, and $n \geq 1$ is a level.

Definition 2.1 (Digit design)**.** A *digit design* is a three-element subset $F \subseteq \{0,1,2\}^2$. Its level-n set is

$$S_n(F) = \Big\{ \textstyle\sum_{i=0}^{n-1} 3^i\, d_i \;:\; d_0, \ldots, d_{n-1} \in F \Big\} \subseteq \mathbf{Z}^2,$$

counted with the digit string that produces it, so that $S_n(F)$ carries 3^n strings. There are $\binom{9}{3} = 84$ digit designs.

Two families matter here, and they are different objects. Confusing them is the one trap in this subject.

Definition 2.2 (The gasket, and the permutation designs)**.** The *gasket design* is $G = \{(0,0),(1,0),(0,1)\}$, and we write $G_n = S_n(G)$; it is the base-3 Sierpiński gasket truncated at level n. For a permutation ϕ of $\mathbf{F}_3$, the *permutation design* is $F_\phi = \{(0,\phi(0)),(1,\phi(1)),(2,\phi(2))\}$. The gasket is not a permutation design and no permutation design is the gasket. Theorem 3.2 is about the F_ϕ; Theorem 3.8 and Fact 3.17 are about G.

Definition 2.3 (Code of a design)**.** Index the nine cells by $3a+b$ for $(a,b) \in \{0,1,2\}^2$. The *code* of a design is $\sum_{(a,b)\in F} 2^{3a+b}$. Under this indexing the four designs of Theorem 3.2 have codes 98 ($\phi = j+1$), 140 ($\phi = j+2$), 266 (ϕ swapping $0,1$) and 84 (ϕ swapping $0,2$). The code 84 is also what the transposed indexing $a+3b$ gives, so no convention rescues the value 148, which names the cell set $\{(0,2),(1,1),(2,1)\}$ and is not a permutation design at all.

Definition 2.4 (Rays, mass, fibres)**.** A *primitive ray* is a pair (a,b) of nonnegative integers, not both zero, with $\gcd(a,b) = 1$; it stands for the half-line $\{z(a,b) : z > 0\}$. The two *fibre rays* are $(1,0)$ and $(0,1)$. For a design F, the *mass* of the ray (a,b) at level n is

$$M_n^F(a,b) = \#\{z \geq 1 \;:\; z\,(a,b) \in S_n(F)\},$$

and we drop the superscript when $F = G$. A ray is *occupied* at level n when its mass is positive. The origin lies on no ray.

Definition 2.5 (Diagonal design, and the count Z_F)**.** A design F is *diagonal* when for every $n \geq 1$ no two distinct nonzero points of $S_n(F)$ are collinear with the origin, that is, when every occupied ray has mass 1. Write $Z_F(n)$ for the number of occupied *non-fibre* rays at level n.

Remark 2.6 (Z is a ray count, not a second moment)*.* $Z_F(n)$ counts occupied non-fibre rays, each once. It is not the second moment $\sum_{(a,b)} M_n^F(a,b)^2$ over non-fibre rays. For a diagonal design the two agree, because every mass is 1; for the gasket they differ wildly, and only the ray count is meant anywhere in this paper.

Definition 2.7 (Multiplier-pair automaton)**.** Fix coprime integers $1 \leq s < t$. The automaton $\mathcal{A}(s,t)$ has states in $\mathbf{Z}^2 \times \mathbf{Z}^2$, start state $(0,0;0,0)$, and from a state $(\mathbf{a};\mathbf{b})$ a digit $d \in G$ is admissible when

$$(s\,d + \mathbf{a}) \bmod 3 \in G \quad \text{and} \quad (t\,d + \mathbf{b}) \bmod 3 \in G,$$

with the reductions taken componentwise; the successor state is then $\big(\lfloor (s\,d+\mathbf{a})/3\rfloor; \lfloor (t\,d+\mathbf{b})/3\rfloor\big)$, again componentwise. Reading the base-3 digits of z from least significant, a path of length n from the start state that returns to the start state records exactly a z with $z, sz, tz \in G_n$. Call a state *live* when it is both reachable from the start state and able to reach the start state; only live states lie on such a path, and there are finitely many of them. Write $T(s,t)$ for the transition matrix carried by the live states, and $\lambda(s,t)$ for its spectral radius. The pair (s,t) is a *shift pair* when $\{s,t\} = \{1, 3^j\}$ for some $j \geq 1$, the case in which multiplication is a digit shift.

Remark 2.8 (Trimming to the live states is not cosmetic)*.* Take $(s,t) = (1,16)$. The automaton reaches eleven states, but only the start state is live: every other reachable state leads away and never comes back. So $T(1,16)$ is the 1×1 matrix (1), carrying the loop on the digit $(0,0)$, and $\lambda(1,16) = 1$. By Proposition 3.16 this says $\#\{z : z, 16z \in G_n\} = 1$ for every n, that is, only $z = 0$ qualifies, which is correct. Had we taken the spectral radius of the whole reachable set we would have got the plastic number $1.3247\ldots$, which measures how fast admissible digit strings accumulate, not how many of them close up into a point. Thirteen of the 829 pairs of Fact 3.17 are affected by the trimming, and all thirteen drop to $\lambda = 1$.

Definition 2.9 (The free-digit automaton)**.** Fix coprime $1 \leq s < t$. The automaton $\mathcal{B}(s,t)$ is built exactly as $\mathcal{A}(s,t)$ in Definition 2.7, with the same states, the same admissibility test and the same successor rule, except that the digit d ranges over all of $\{0,1,2\}^2$ and not only over G. Write $T^\flat(s,t)$ for its live transition matrix and $\lambda^\flat(s,t)$ for the spectral radius. Reading the base-3 digits of z from least significant, a closed path of length n at the start state records exactly a $z \in [0, 3^n)^2$ with $sz, tz \in G_n$; the vector z itself is no longer required to lie in G_n. The proof of Proposition 3.16 applies verbatim, so $\#\{z : sz, tz \in G_n\} = ((T^\flat)^n)_{00}$.

Lemma 2.10 (The two automata coincide when a multiplier is one)**.** *For every t one has $\mathcal{B}(1,t) = \mathcal{A}(1,t)$, with the same states and the same edges.*

Definition 2.11 (Collinear pair count)**.** For the gasket put

$$E(n) = \sum_{(a,b)\ \text{non-fibre}} M_n(a,b)^2,$$

the number of ordered pairs of nonzero non-fibre points of G_n lying on a common ray. This is a second moment, not the ray count $Z_F(n)$ of Definition 2.5; Remark 2.6 is the warning, and the two are different numbers for the gasket. Split $E(n) = D(n)+\Sigma(n)+R(n)$: the *diagonal* part $D(n)$ counts the pairs with both entries equal, the *shift-multiplier* part $\Sigma(n)$ counts the ordered pairs (x,y) with $x \neq y$ whose reduced multiplier ratio is $3^{\pm j}$, and the *residual* $R(n)$ is everything else.

Definition 2.12 (The witness of a collinear pair)**.** Let $x \neq y$ be nonzero non-fibre points of G_n on a common ray, with primitive direction r and $x = g_1 r$, $y = g_2 r$. Put $d = \gcd(g_1, g_2)$, $s = g_1/d$, $t = g_2/d$ and $z = dr$. Then $\gcd(s,t) = 1$, $z \in \mathbf{Z}^2_{\geq 1}$, and $(x,y) = (sz, tz)$. Conversely every triple (s,t,z) with $\gcd(s,t) = 1$ and $sz, tz \in G_n$ arises from exactly one such pair, so the assignment is a bijection. Call z the *witness* and $w = z_1 + z_2$ its *weight*, and write

$$M_n(z) = \#\{m \geq 1 : mz \in G_n\},$$

which for primitive z is the ray mass $M_n(z_1, z_2)$ of Definition 2.4. Grouping the residual by its witness gives

$$R(n) = \sum_{z \in \mathbf{Z}^2_{\geq 1}} P_n(z),$$

$$P_n(z) = \#\{(s,t) : \gcd(s,t) = 1,\ s \neq t,\ t/s \neq 3^{\pm j},\ sz, tz \in G_n\},$$

and $P_n(z) \leq M_n(z)^2 - M_n(z)$. Write $R_w(n)$ for the part of $R(n)$ carried by witnesses of weight w.

3. Results

3.1. **Law 1: which permutation designs are diagonal.**

Lemma 3.1 (Cross lemma)**.** *Let ϕ be a permutation of $\mathbf{F}_3$ and let $x, x' \in S_n(F_\phi)$ come from distinct digit strings $d \neq d'$, first differing at position k. Then*

$$\det(x,x') \;=\; x_1 x_2' - x_2 x_1' \;\equiv\; 3^k\, \phi(0)\, (d_k - d_k') \pmod{3^{k+1}}.$$

In particular, if $\phi(0) \neq 0$ then $v_3(\det(x,x')) = k$ exactly, so $\det(x,x') \neq 0$.

Theorem 3.2 (Diagonal classification)**.** *Let ϕ be a permutation of $\mathbf{F}_3$. Then F_ϕ is a diagonal design if and only if $\phi(0) \neq 0$. Four of the six permutations qualify: $j \mapsto j+1$, $j \mapsto j+2$, the swap of 0 and 1, and the swap of 0 and 2. For each of these four and every $n \geq 1$, the 3^n level-n digit strings give*

3^n distinct nonzero points on 3^n distinct rays, exactly two of which are fibre rays, so

$$Z_{F_\phi}(n) = 3^n - 2 \qquad (n \geq 1).$$

For the two permutations with $\phi(0) = 0$ the design is not diagonal: the identity fails at level 1, where $(1,1)$ and $(2,2)$ share a ray, and $\phi(j) = 2j$ fails at level 2, where $(1,2)$ and $(3,6)$ share a ray.

Remark 3.3 (The bound starts at $n = 1$). The formula $Z_{F_\phi}(n) = 3^n - 2$ is false at $n = 0$: the empty digit string gives the origin, which lies on no ray, so $Z_{F_\phi}(0) = 0$ while $3^0 - 2 = -1$. The statement is scoped to $n \geq 1$ throughout.

Diagonality by itself, without the permutation hypothesis, is a slightly weaker condition, and the following census says exactly how much weaker.

Fact 3.4 (Census of the 84 designs)**.** Among the 84 digit designs, exactly 10 have the property that for every level $n \leq 6$ no two distinct nonzero points of $S_n(F)$ are collinear with the origin. Exactly 4 of those 10 are permutation designs, namely the four of Theorem 3.2. The other six are

$$\{(0,1),(1,1),(2,1)\}, \quad \{(0,2),(1,2),(2,2)\}, \quad \{(1,0),(1,1),(1,2)\},$$
$$\{(1,1),(1,2),(2,1)\}, \quad \{(1,2),(2,1),(2,2)\}, \quad \{(2,0),(2,1),(2,2)\};$$

they are not permutation designs, and while four of them do have a constant coordinate, the two designs $\{(1,1),(1,2),(2,1)\}$ and $\{(1,2),(2,1),(2,2)\}$ do not. The four permutation designs are also exactly the survivors that occupy two fibre rays; the other six occupy one or none. Hence the sharp criterion inside the census is "no two points collinear, and exactly two fibre rays occupied". Checked by `scripts/verify.py` over all 84 designs and levels $1 \leq n \leq 6$.

3.2. **Law 2: how much mass each gasket ray carries.** The gasket is the opposite of diagonal, so masses are the interesting invariant. One global count is immediate and exact at every level.

Proposition 3.5 (Total non-fibre mass)**.** *For every $n \geq 0$, the number of nonzero level-n gasket points lying off both fibre rays is*

$$\sum_{(a,b)\ \text{non-fibre}} M_n(a,b) \;=\; 3^n - 2^{n+1} + 1.$$

At $n = 12$ this is $531441 - 8192 + 1 = 523250$.

Definition 3.6 (Ray carry automaton)**.** For a primitive ray (a,b) let $\mathcal{C}(a,b)$ have states in $\mathbf{Z}$, start state 0, and from a state c a digit $d = (d_1, d_2) \in G$ admissible when $3 \mid c + b\,d_1 - a\,d_2$, with successor $(c + b\,d_1 - a\,d_2)/3$. Write $T(a,b)$ for the live transition matrix.

Proposition 3.7 (The carry automaton counts the ray)**.** *For every primitive ray (a,b) and every $n \geq 0$, $M_n(a,b) = (T(a,b)^n)_{00} - 1$, where $T(a,b)$ is as in Definition 3.6 and the start state carries index 0; the path removed* *is the all-$(0,0)$ one, which is the origin. Every reachable state satisfies $-a/2 \le c \le b/2$, so the live set has at most $\lfloor a/2\rfloor + \lfloor b/2\rfloor + 1$ elements, half the weight $a+b$ and one more.*

Theorem 3.8 (Ray mass laws)**.** *Let $F(1) = F(2) = 1$ be the Fibonacci numbers [3], let $a(0) = a(1) = a(2) = 1$ and $a(m) = a(m-1) + a(m-3)$ be Narayana's cows [4], and let $c(0), c(1), c(2), c(3) = 1, 2, 3, 4$ with $c(m) = c(m-1) + c(m-4)$ for $m \ge 4$. Then for every level:*

(1) $M_n(3,1) = F(n+1) - 1$ *for* $n \ge 1$*;*
(2) $M_n(3^j,1) = M_n(1,3^j) = \left(\prod_{r=0}^{j-1} F(m_r+2)\right) - 1$ *for all* $j \ge 1$ *and* $n \ge 1$*, where* $m_r = \#\{i \in [0, n-j) : i \equiv r \bmod j\}$*; equivalently, writing* $n - j = qj + \rho$ *with* $0 \le \rho < j$*, the product is* $F(q+3)^{\rho} F(q+2)^{j-\rho}$*;*
(3) $M_n(1,12) = a(n) - 1$ *for* $n \ge 1$*;*
(4) $M_n(7,3) = c(n-3) - 1$ *for* $n \ge 3$*.*

The three named recurrences are exactly the characteristic polynomials of three small live automata, and the shift family is a run-length count in disguise; both routes appear in Section 4. Independently, `scripts/verify.py` cross-checks the automaton against enumeration of the gasket points themselves: directly over all 3^n points for $1 \le n \le 10$, and for $1 \le n \le 14$ by splitting each point as $u + 3^{\lfloor n/2\rfloor} v$ with u, v in gasket sets of half the level, which counts the same 3^n points without listing them.

Theorem 3.9 (The shift family carries φ^{2n} and no more)**.** *Let $\varphi = (1+\sqrt{5})/2$ and put*

$$\mathrm{Sh}(n) = \sum_{j\ge 1} \left(M_n(3^j,1)^2 + M_n(1,3^j)^2\right),$$

the part of $E(n)$ carried by the shift rays. Then for every $n \ge 1$ and every $j \ge 1$,

$$M_n(3^j,1) < (3-\sqrt{5})^j\,\varphi^n, \qquad \mathrm{Sh}(n) < \frac{4+12\sqrt{5}}{11}\,\varphi^{2n} < 2.803\,\varphi^{2n},$$

and

$$\lim_{n\to\infty} \frac{\mathrm{Sh}(n)}{\varphi^{2n}} = \frac{13+5\sqrt{5}}{11} = 2.198212717\ldots$$

The sum over j is finite at each level, since $M_n(3^j,1) = 0$ for $j \ge n$.

Proposition 3.10 (The shift-multiplier part in closed form)**.** *For $1 \le j \le n-1$ the number of nonzero non-fibre z with $z, 3^j z \in G_n$ is $3^{n-j} - 2^{n-j+1} + 1$, and therefore*

$$\Sigma(n) = 2\sum_{j=1}^{n-1} \left(3^{n-j} - 2^{n-j+1} + 1\right) = 3^n - 4\cdot 2^n + 2n + 3.$$

With Proposition 3.5 this gives $D(n) + \Sigma(n) = 2\cdot 3^n - 6\cdot 2^n + 2n + 4$ exactly, so the whole question of whether $E(n)/3^n$ converges, and to what, is the question of how $R(n)$ grows.

Remark 3.11 (The seeds are $1,2,3,4$, not $1,1,1,1$). The $(7,3)$ recurrence must be started at $c(0), c(1), c(2), c(3) = 1,2,3,4$. Starting it at $1,1,1,1$ produces a different sequence and the identity in Theorem 3.8(4) fails immediately.

Remark 3.12 (Squares on the ray $(7,3)$). $M_{14}(7,3) = 49 = 7^2$ is not singular. The mass is a perfect square at exactly $n = 4,7,9,12,14$ in the range $1 \le n \le 30$, with values $1,4,9,25,49$, and at no n in $15 \le n \le 30$. There is also no algebraic reason to expect squares: the characteristic polynomial $x^4 - x^3 - 1$ of the recurrence is irreducible over $\mathbf{Q}$. Indeed it has no rational root, since by the rational root theorem the only candidates are ± 1 and neither is a root; and by Gauss's lemma a factorisation into two rational quadratics may be taken integral, $(x^2 + ax + b)(x^2 + cx + d)$, which forces $bd = -1$, so $\{b, d\} = \{1, -1\}$ and the linear coefficient $ad + bc = \pm(a - c)$ vanishes only if $a = c$, contradicting $a + c = -1$ in the integers.

Fact 3.13 (The level-12 census)**.** At level $n = 12$ the gasket has $3^{12} = 531441$ points. Exactly 345318 non-fibre primitive rays are occupied, and they carry 523250 points in total, in agreement with Proposition 3.5. Ray mass is symmetric under $x \leftrightarrow y$, so occupied rays come in equal-mass mirror pairs; the ten heaviest are five shift rays and their five mirrors:

ray	$(3,1)$	$(9,1)$	$(27,1)$	$(81,1)$	$(243,1)$
mass	232	168	124	80	71

and $232 = F(13) - 1$. Checked by `scripts/verify.py` by literal enumeration of all 531441 points, with the total mass identity of Proposition 3.5 rechecked at every level $1 \le n \le 12$.

Fact 3.14 (The second moment to level 17)**.** For $1 \le n \le 12$ the collinear pair count of Definition 2.11 reads

$$\begin{aligned} E(n) = 0,\ 2,\ 16,\ 98,\ 396,\ 1522,\ 5248,& \\ 17118,\ 52212,\ 158042,\ 466960,\ 1374038,& \end{aligned}$$

and the residual $R(n) = E(n) - D(n) - \Sigma(n)$ reads

$$R(n) = 0,\ 0,\ 0,\ 20,\ 88,\ 432,\ 1624,\ 5512,\ 15896,\ 46064,\ 124928,\ 335704.$$

At every one of those levels the shift-ray share $\mathrm{Sh}(n)$ agrees with the closed form of Theorem 3.8(2), and it is the dominant part of R: the ordered off-diagonal pairs carried by shift rays but not by shift multipliers number 360, 1204, 3816, 10656, 30132, 81960, 221980 at $n = 6, \dots, 12$, a share of $R(n)$ that stays between 0.65 and 0.84 and reads 0.661 at $n = 12$. Checked by `scripts/verify.py` by literal enumeration of all 3^n points.

Five further levels, computed by grouping all 3^n points by primitive direction and taking $E(n) = \sum_r M_n(r)^2$, then subtracting $D(n) = 3^n - 2^{n+1} + 1$

and Proposition 3.10, continue for $n = 13, \dots, 17$ with

$$E(n) = 4003372,\ 11679626,\ 34050692,\ 99800950,\ 292848756,$$
$$R(n) = 863848,\ 2211960,\ 5549452,\ 14100688,\ 35354824.$$

Two ratios matter. $R(n)/3^n$ peaks at 0.84012 at $n = 8$ and then falls at every level to $0.2737709\dots$ at $n = 17$. $R(n)/\varphi^{2n}$ peaks at 3.2378 at $n = 12$ and then falls at every level to $2.7724831\dots$ at $n = 17$, and the level ratio $R(n+1)/R(n)$ has settled to $2.5073119\dots$ at $n = 17$, below $\varphi^2 = 2.6180\dots$ So the residual now looks like a φ^{2n} quantity with a constant below 3.24, not like a 3^n quantity.

Fact 3.15 (The level-9 multiplier census)**.** At level $n = 9$ the gasket occupies 12170 non-fibre primitive rays carrying 18660 points, and

$$E(9) = 52212, \quad D(9) = 18660, \quad \Sigma(9) = 17656, \quad R(9) = 15896.$$

The 33552 ordered off-diagonal collinear pairs are carried by exactly 2656 ordered coprime multiplier pairs (s, t), that is 1328 unordered ones, of which 14 are shift pairs, namely $(1, 3^j)$ and $(3^j, 1)$ for $1 \le j \le 7$; the largest multiplier occurring is 2460. For each of the 1328 unordered pairs the return counts of $\mathcal{A}(s, t)$ agree with brute-force enumeration.

They agree with a count that is not the census summand. Writing each off-diagonal collinear pair as (sz, tz) with $\gcd(s, t) = 1$, the automaton $\mathcal{A}(s, t)$ counts only those witnesses with $z \in G_n$, whereas the pair exists for every integer z. Of the 2656 ordered pairs, 482 have a witness off the gasket, and 2540 of the 33552 ordered pairs, that is 7.57%, are missed. Every one of the 482 has $\min(s, t) \ge 2$, in agreement with Lemma 2.10; the extreme cases are $(41, 122)$, $(122, 41)$, $(122, 123)$ and $(123, 122)$, each with 50 census witnesses and none of them in G_9. The free-digit automaton $\mathcal{B}(s, t)$ of Definition 2.9 counts the census summand itself, and its return counts agree with brute force on all 103 unordered pairs with $\max(s, t) \le 52$, which carry 11330 of the 16776 unordered off-diagonal pairs.

3.3. **Law 3: how fast a pair of multiples can grow.**

Proposition 3.16 (The radius is the growth rate)**.** *Let $1 \le s < t$ be coprime and let $T = T(s, t)$ be the live transition matrix of Definition 2.7, with the start state carrying index 0. Then for every $n \ge 0$*

$$\#\{z : z, sz, tz \in G_n\} = (T^n)_{00},$$

and consequently

$$\lim_{n\to\infty} \#\{z : z, sz, tz \in G_n\}^{1/n} = \lambda(s, t).$$

Fact 3.17 (Spectral gap at two)**.** Let (s, t) range over the 829 coprime unordered pairs with $1 \le s < t \le 52$. Then:

(1) $\lambda(s, t) = 3$ for exactly three pairs, namely the shift pairs $(1, 3)$, $(1, 9)$, $(1, 27)$;
(2) no pair has $\lambda(s, t)$ in the open interval $(2, 3)$;

(3) $\lambda(s,t) = 2$ for exactly twenty pairs, listed in Table 1;
(4) the largest value strictly below 2 is $\lambda = 1.6956207695598\ldots$, attained by 25 pairs, the smallest being $(1,13)$ and including $(12,13)$; the next value down is $1.6769719158912\ldots$, attained by $(1,31)$, $(3,31)$, $(9,31)$ and $(27,31)$;
(5) the pairs $(2,9)$ and $(1,18)$, whose multipliers are divisible by 9 after reordering, have $\lambda = 1$: their live sets are the start state alone.

Checked by `scripts/verify.py`, which builds each automaton from Definition 2.7, trims it to its live states, forms the characteristic polynomial by Faddeev–LeVerrier in exact integer arithmetic, and certifies the absence of a real root above a rational r by shifting the polynomial to r and observing that every coefficient is nonnegative. Because the transition matrix is nonnegative, its spectral radius is itself a real eigenvalue, so this certificate is a proof of $\lambda \le r$ for each individual pair. Every quoted decimal is pinned from both sides in exact rational arithmetic: the upper side by that certificate, the lower side by a rational point at which the characteristic polynomial is negative, which forces a real root above it.

$(1,4)$	$(1,7)$	$(1,10)$	$(1,12)$	$(1,21)$	$(1,28)$
$(1,30)$	$(1,36)$	$(3,4)$	$(3,7)$	$(3,10)$	$(3,28)$
$(4,9)$	$(4,27)$	$(7,9)$	$(7,27)$	$(9,10)$	$(9,28)$
$(10,27)$	$(27,28)$				

Table 1. The twenty coprime pairs with $\max(s,t) \le 52$ attaining $\lambda = 2$ exactly.

Fact 3.18 (The free-digit automaton keeps the gap but not the ceiling)**.** Over the same 829 pairs, $\lambda^\flat(s,t)$ obeys the first three items of Fact 3.17 exactly: it equals 3 on exactly the three shift pairs $(1,3)$, $(1,9)$, $(1,27)$; it never lies in the open interval $(2,3)$; and it equals 2 on exactly the twenty pairs of Table 1. Item (4) does *not* transfer. The largest value of $\lambda^\flat$ strictly below 2 is

$$1.8488475886485\ldots,$$

attained by exactly four pairs, $(4,13)$, $(4,39)$, $(12,13)$ and $(13,36)$. Below 2 the gasket-digit ceiling $\theta = 1.6956207695598\ldots$ of Fact 3.17(4), the real root of $x^3 - x^2 - 2$, is reached or beaten by 44 pairs, split sharply: exactly 19 have $\lambda^\flat$ strictly above θ, and exactly 25 more have $\lambda^\flat = \theta$ exactly, their characteristic polynomials carrying $x^3 - x^2 - 2$ as a factor while none of the 19 does. A witness that needs no eigenvalue: $\mathcal{B}(4,13)$ has 4583352807133551 closed paths of length 60 at the start state, against $1.6956207695598\ldots^{60} < 5.76 \cdot 10^{13}$. The automata are larger too, up to 167 live states, attained at both $(25,52)$ and $(31,40)$, against 33 for $\mathcal{A}$, though $\mathcal{B}(1,16)$ still trims to the start state alone. Checked by the same exact integer certificates as

Fact 3.17. So what survives the correction of Fact 3.15 is the gap itself, on the quantity the pair census actually needs; the sharp constant below 2 does not.

The simplest attainer can be checked by hand, so we prove it.

Proposition 3.19 (The pair $(1,4)$)**.** *The automaton $\mathcal{A}(1,4)$ has exactly three reachable states, all of them live: $A=(0,0;0,0)$, $B=(0,0;1,0)$ and $C=(0,0;0,1)$, with transitions $A\to A$, $A\to B$, $A\to C$, $B\to A$, $C\to A$. Its characteristic polynomial is $\lambda^3-\lambda^2-2\lambda=\lambda(\lambda-2)(\lambda+1)$, so $\lambda(1,4)=2$. The number G_m of paths of length m that start at A, ending anywhere, satisfies*

$$G_m = G_{m-1} + 2G_{m-2} \quad (m \geq 2), \qquad G_0 = 1,\ G_1 = 3,$$

whence $G_m = (2^{m+2}-(-1)^m)/3$, giving 1, 3, 5, 11, 21, 43, 85, 171, *and* $G_{15}=43691$. *The arithmetic count is the number $R_m=(T^m)_{AA}$ of paths that return to A, which obeys the same recurrence from $R_0=R_1=1$, so*

$$\#\{z : z, 4z \in G_m\} = R_m = (2^{m+1}+(-1)^m)/3,$$

giving 1, 1, 3, 5, 11, 21, 43, 85, 171, *and* $R_{15}=21845$. *Both grow like 2^m, but only R_m counts points.*

3.4. **Law 4: what a witness can weigh.** The multiplier pair is the wrong coordinate on the residual. Remark 3.45 explains why: the active pairs already number about 2.77^n, so a per-pair bound is useless without control of a sum of constants over that set. The witness of Definition 2.12 is the right coordinate, because the sum over witnesses is indexed by a set on which the ray mass law of Theorem 3.8 already speaks.

Lemma 3.20 (No witness weighs less than four)**.** *Let $z\in\mathbf{Z}^2_{\geq 1}$ and $m\geq 1$ with $mz\in G_n$. Then $z_1+z_2\geq 4$. Consequently every multiplier pair active at level n satisfies*

$$\max(s,t) \;\leq\; \frac{3^n-1}{8},$$

whose floor is $\lfloor 3^n/8\rfloor$, and that is exactly the largest multiplier occurring, at every level $4\leq n\leq 13$. The bound is therefore sharp.

Proposition 3.21 (Large multipliers contribute exactly four)**.** *Let $\{s,t\}$ be a coprime non-shift pair with $\max(s,t)>(3^n-1)/10$. Then every witness of the pair has weight exactly* 4 *and lies in $\{(1,3),(3,1)\}$, both occur or neither, and the pair carries exactly* 4 *ordered off-diagonal collinear pairs of G_n, or none. At $n=6,\ldots,13$ the pairs above that threshold number* 18, 57, 163, 402, 1019, 2702, 7060, 18607, *and every one of them carries exactly* 4.

Proposition 3.22 ($\mathcal{B}$ is a constrained tensor square)**.** *Fix coprime $s<t$. Let $\mathcal{C}(s,t)$ be the* carry automaton *on states $(a,b)\in\mathbf{Z}^2_{\geq 0}$, start $(0,0)$, carrying for each digit $d\in\{0,1,2\}$ with $u=(sd+a)\bmod 3\leq 1$ and $v=(td+b)\bmod 3\leq 1$ an edge to $(\lfloor(sd+a)/3\rfloor,\lfloor(td+b)/3\rfloor)$ labelled (u,v). Let S be its transition matrix, and U, V, W the matrices of the edges with $u=1$,*

with $v = 1$, and with $u = v = 1$. Identify a state $(\mathbf{a};\mathbf{b})$ *of* $\mathcal{B}(s,t)$ *with the pair of* $\mathcal{C}$*-states* $\big((a_1,b_1),(a_2,b_2)\big)$. *Then*

$$T^\flat(s,t) \;=\; S\otimes S - U\otimes U - V\otimes V + W\otimes W.$$

Hence $\mathcal{C}$ *has at most* $(s+1)(t+1)$ *reachable states, and the return counts of Definition 2.9 are produced by* n *steps of* $X \mapsto SXS^\top - UXU^\top - VXV^\top + WXW^\top$ *from* $X = e_0e_0^\top$, *reading* X_{00}, *with no four-tuple state graph built. On all* 473 *coprime pairs below* 40 *the two agree at every level to* 9. *The carry automaton is the smaller object:* 729 *live carry states against* 26931 *reachable states of* $\mathcal{B}$ *at* $(365,1094)$, *and* 81 *against* 835 *at* $(41,122)$.

Proposition 3.23 (The box construction). *Fix coprime* $s < t$ *and put* $W_n(t) = \lfloor (3^n-1)/(2t)\rfloor$. *Then*

$$\begin{aligned} N_n(s,t) &:= \#\{z \in \mathbf{Z}^2_{\ge 1} : sz, tz \in G_n\} \\ &= \#\{z \in \mathbf{Z}^2_{\ge 1} : z_1 + z_2 \le W_n(t),\; sz, tz \in G_n\}, \end{aligned}$$

The two symbols are duals and must not be confused: $N_n(s,t)$ *counts the witnesses of one pair, while* $P_n(z)$ *of Definition 2.12 counts the pairs of one witness. So* $N_n(s,t)$ *is computed by* $O(W_n(t)^2\,n)$ *digit tests and no automaton at all. It agrees with* $\mathcal{B}(s,t)$ *on all* 812 *coprime pairs with* $s < 30$, $t < 60$ *at* $n = 9$, *and with Proposition 3.22 at* $(365,1094)$, $(41,122)$, $(122,123)$, $(1,2460)$ *and* $(2431,2458)$ *at* $n = 9$ *and* $n = 12$. *The construction is exactly complementary to the automaton: it is cheapest where a forward build of* $\mathcal{B}$ *is most expensive.*

Theorem 3.24 (The weight-four layer is Fibonacci and closed). *Let* $F_n = \{m \ge 1 : (m,3m) \in G_n\}$, *which is the set of* $m < 3^{n-1}$ *whose base-*3 *expansion is binary with no two adjacent ones, so* $\#F_n = F(n+1) - 1$. *Then*

$$\begin{aligned} R_4(n) &= 2\,\#\{(a,b) \in F_n^2 : a \ne b,\; \gcd(a,b) = 1,\; b/a \ne 3^{\pm j}\} \\ &<\; 2\big(F(n+1)-1\big)^2 \;<\; 1.0473\,\varphi^{2n}, \end{aligned}$$

and the weight layers scale exactly as $R_{3w}(n) = R_w(n-1)$, *so the whole* 3*-power orbit of weight four obeys* $\sum_{j\ge 0} R_4(n-j) < 1.6945\,\varphi^{2n}$. *The identity is checked at* $n = 4,\dots,12$, *where* $R_4(n) = 12$, 36, 108, 336, 988, 2596, 6672, 17480, 45720*; the scaling law is checked on all* 1869 *layers of weight divisible by* 3 *at* $n = 5,\dots,13$. *At* $n = 13$ *the weight-four orbit carries* 194096 *of* $R(13) = 863848$.

Lemma 3.25 (Branch structure of a ray automaton). *Fix a primitive ray* (a,b) *with* $a, b \ge 1$, *so that* 0, b, $-a$ *are three distinct values. As* d *runs over* G *the increment of Definition 3.6 takes exactly those values, so a state* c *of* $\mathcal{C}(a,b)$ *has one outgoing edge for each* $\varepsilon \equiv -c \pmod 3$. *Then:*

(1) *at most one of* a, b, $a+b$ *is divisible by* 3*;*

(2) *if* $3 \nmid ab$ *then the only closed path at* 0 *is the all-*$(0,0)$ *one and* $M_n(a,b) = 0$ *for every* n*, whether or not* 3 *divides* $a+b$*; so* 3 *divides a coordinate of every occupied direction;*

(3) *otherwise let* $q \in \{a,b\}$ *be the coordinate divisible by* 3 *and* $k = v_3(q) \geq 1$*. The congruent pair of increments is* $\{0,b\}$ *or* $\{0,-a\}$ *and its two members differ by* $\pm q$*, so every state has out-degree at most two, and in the untrimmed automaton the states of out-degree two, the* branch *states, are exactly the states divisible by* 3*. The two successors of a branch state differ by* $\pm q/3$*, so they lie in different classes modulo* 3 *when* $k = 1$ *and in the same class when* $k \geq 2$*. Trimming to the live set only deletes successors, never adds them, and a deleted successor is one that cannot reach the start state, so it contributes* 0 *to every path count below; every bound proved for the untrimmed automaton therefore holds for the trimmed one, while a state's out-degree may drop.*

Lemma 3.26 (Occupancy needs a residue match)**.** *Let* z *be primitive with* $z_1, z_2 \geq 1$ *and* $3 \mid z_1 z_2$*. Write* $\{z_1, z_2\} = \{p, q\}$ *with* $3 \mid q$*, and* $q = 3^k q_1$ *with* $k = v_3(q)$ *and* $3 \nmid q_1$*. If* $M_n(z) > 0$ *for some* n *then*

$$q_1 \equiv p \pmod 3.$$

Remark 3.27 (What the residue match removes). The condition is a congruence on the pair, tested without building an automaton. Of the 11691 primitive z with $3 \mid z_1 z_2$ in the box and the six families of Fact 3.40, exactly 7103 match and 4588 do not, and every one of the 865 occupied directions matches. It is only necessary: just 865 of the 7103 matching directions carry any mass. Lemma 3.25(2) removes the directions with $3 \nmid z_1 z_2$ and Lemma 3.26 removes 4588 of the 11691 left, both by arithmetic alone and neither needing an automaton.

Theorem 3.28 (The golden ceiling, proved cases)**.** *Let* (a,b) *be primitive with* $a, b \geq 1$*. Then* $M_n(a,b) \leq F(n+1) - 1$ *for every* $n \geq 0$ *in each of these cases.*

(1) $3 \nmid ab$*, where in fact* $M_n(a,b) = 0$*.*

(2) *No branch state of* $\mathcal{C}(a,b)$ *has both successors branching, which by Lemma 3.25(3) holds whenever* $k = v_3(q) = 1$*.*

(3) $\{a,b\} = \{1, 3^j\}$*, and there the bound is strict for* $j \geq 2$ *and* $n \geq 2$*.*

(4) $\mathcal{C}(a,b)$ *carries a* Fibonacci certificate*: rationals* $\alpha(c) \geq 0$ *and* $\beta(c)$ *on the live states, with* $\alpha(0) = 1$*,* $\beta(0) = 0$ *and*

$$\sum_{c'} \alpha(c') \leq \alpha(c) + \beta(c), \qquad \sum_{c'} \beta(c') \leq \alpha(c)$$

at every state c*, both sums running over the successors of* c *with multiplicity.*

Remark 3.29 (Both hypotheses are load-bearing). The restriction $a, b \geq 1$ in Lemma 3.25 is not decorative, and neither is counting edges with multiplicity.

On the fibre ray $(a,b) = (0,1)$ the increments 0, b, $-a$ are 0, 1, 0: the digits $(0,0)$ and $(0,1)$ give the same increment, so the start state carries two edges, both to itself, and $(T^n)_{00} = 2^n$. Read as a set of successor states the automaton would show no branch state at all, and Theorem 3.28(2) would return the ceiling; but $M_n(0,1) = 2^n - 1$, which is 63 against $F(7) - 1 = 12$ at $n = 6$. For $a, b \geq 1$ the three increments are distinct and the reading is unambiguous.

Remark 3.30 (Why case (2) does not extend). Case (2) is proved by bounding $G(n) = \max_c N(c,n)$, where $N(c,n)$ counts the paths of length n from c to the start state, and the proof needs $G(n) \leq G(n-1) + G(n-2)$. That inequality is false in general. At $z = (1,9)$ the profile runs $G(0), \ldots, G(6) = 1$, 1, 1, 2, 4, 6, 9 and already $G(4) = 4 > G(3) + G(2) = 3$; over the box of Proposition 3.38 it fails for eight directions, all with $k \geq 2$. So $k \geq 2$ needs a different mechanism, and cases (3) and (4) are two of them.

Definition 3.31 (First returns and the golden potential)**.** Let (a,b) be primitive with $a, b \geq 1$. For a live state c of $\mathcal{C}(a,b)$ let $g(c,m)$ count the paths of length m from c to the start state that meet the start state only at their last step, with $g(0,0) = 1$ and $g(0,m) = 0$ for $m \geq 1$, and set

$$u(c) = \sum_{m \geq 0} g(c,m) \varphi^{-m} \in [0, \infty], \qquad U(a,b) = \sum_{c' \neq 0} u(c'),$$

the second sum running over the successors of the start state other than the start state, with multiplicity. Write f_j for the number of closed paths of length j at the start state that meet it only at their two ends, so $f_1 = 1$, $f_j = \sum_{c' \neq 0} g(c', j-1)$ for $j \geq 2$, and $\sum_{j \geq 2} f_j \varphi^{-j} = \varphi^{-1} U(a,b)$.

Theorem 3.32 (The golden ceiling from a potential)**.** *Let (a,b) be primitive with $a, b \geq 1$, and suppose there is $\pi > 0$ on the live states of $\mathcal{C}(a,b)$ with*

$$\sum_{c'} \pi(c') \leq \varphi\, \pi(c) \ \text{ for every live } c \neq 0, \qquad \sum_{c' \neq 0} \pi(c') \leq \varphi^{-2} \pi(0),$$

both sums over successors with multiplicity, the second over the successors of the start state other than the start state. Then $U(a,b) \leq \varphi^{-2}$ and $M_n(a,b) \leq F(n+1) - 1$ for every $n \geq 0$. Conversely $\pi = u$ is such a potential as soon as $U(a,b) \leq \varphi^{-2}$, so the hypothesis says exactly that.

Remark 3.33 (What the criterion misses, and why $(1,3)$ is extremal). On the shift ray $\{a,b\} = \{1, 3^j\}$, Theorem 3.8(2) writes $N(0,n) = M_n + 1$ as $\prod_{r<j} F(m_r + 2)$ with $m_r \geq 0$ and $\sum_r m_r = n - j$, so $N(0,n) \geq \varphi^{n-j}$ for $n > j$ by the envelope $F(m) \geq \varphi^{m-2}$ in the proof of Theorem 3.32, and $\sum_n N(0,n) x^n = 1/(1 - \Phi(x))$, with $\Phi(x) = \sum_{j \geq 1} f_j x^j$, blows up as x rises to φ^{-1}; since Φ has nonnegative coefficients, monotone convergence gives $\Phi(\varphi^{-1}) = 1$, that is $U(1, 3^j) = \varphi(1 - \varphi^{-1}) = \varphi^{-1}$ exactly. So Theorem 3.32 misses every shift ray, and Theorem 3.28(3) is exactly the complementary tool. Length-two first returns say why $(1,3)$ is the heaviest ray: f_2 counts the pairs of increments $\varepsilon_0 \neq 0$ with $3 \mid \varepsilon_0$ and $\varepsilon_1 = -\varepsilon_0/3$, and $\varepsilon_0 = b$ forces $b = 3a$ while $\varepsilon_0 = -a$ forces $a = 3b$, so $f_2 = 1$ when $\{a, b\} = \{1, 3\}$ and $f_2 = 0$ at every other direction.

Lemma 3.34 (Short first returns)**.** *Let z be primitive with $z_1, z_2 \geq 1$ and $3 \mid z_1 z_2$, and let p, $q = 3^k q_1$ be as in Lemma 3.26. Then:*

1. *$f_j = 0$ for $2 \leq j \leq k$, so the shortest first return through a state other than the start state has length more than $v_3(q)$;*
2. *$f_2 = 1$ if $\{z_1, z_2\} = \{1, 3\}$ and $f_2 = 0$ at every other direction;*
3. *$f_3 = 1$ if $\{z_1, z_2\}$ is one of $\{1, 9\}$, $\{1, 12\}$, $\{3, 10\}$, $\{4, 9\}$, and $f_3 = 0$ at every other direction;*
4. *away from $\{1, 3\}$,*
$$U(z) = \varphi^{-2} \sum_{j \geq 3} f_j \varphi^{3-j},$$
so $U(z) \leq \varphi^{-2}$ says exactly $\sum_{j \geq 3} f_j \varphi^{3-j} \leq 1$. Away from the four directions of (3) this forces $f_4 \leq 1$, and $f_4 = 1$ forces $\sum_{j \geq 5} f_j \varphi^{5-j} \leq 1$ in turn.

Theorem 3.35 (The degree potential and its sweep)**.** *Let (a, b) be primitive with $a, b \geq 1$ and occupied. Put $\pi_0(0) = 1$ and, at a live state $c \neq 0$, put $\pi_0(c) = 1$ if its live out-degree is two and $\pi_0(c) = \varphi^{-1}$ if it is one. Define $\pi_{d+1}(0) = 1$ and $\pi_{d+1}(c) = \varphi^{-1} \sum_{c'} \pi_d(c')$ at live $c \neq 0$, the sum over successors with multiplicity. If π_0 satisfies the first inequality of Theorem 3.32, which happens whenever no branch state of the live automaton has both successors of out-degree two and is in any case a finite check, then*

$$u \leq \pi_{d+1} \leq \pi_d \leq \pi_0 \quad \textit{pointwise, so} \quad U(a, b) \leq \sum_{c' \neq 0} \pi_d(c') \quad \textit{for every } d \geq 0,$$

a decreasing sequence of bounds in $\mathbf{Q}(\sqrt{5})$; if one of them is at most φ^{-2} then so is $U(a, b)$, and Conjecture 3.41 holds at that direction.

Theorem 3.36 (The golden partition bound where 3 divides a coordinate exactly once)**.** *Let z be primitive and occupied with $z_1, z_2 \geq 1$, let p and $q = 3q_1$ be as in Lemma 3.26 with $v_3(q) = 1$, and suppose $\{z_1, z_2\} \neq \{1, 3\}$. Then $t := v_3(q_1 - p)$ is finite and at least 1, and*

$$U(z) \ \leq \ \varphi^{-1}\big(1 - \varphi^{-\max(t,2)}\big) \ < \ \varphi^{-1}.$$

In particular $U(z) \leq \varphi^{-2}$ whenever $t \leq 2$, so Conjecture 3.41 and with it the golden ceiling at every level hold for every primitive direction in the arithmetically defined class

$$v_3(q) = 1, \qquad v_3(q/3 - p) \leq 2, \qquad \{z_1, z_2\} \neq \{1, 3\},$$

an infinite family, in which the members with $t = 0$ are covered too: they fail the residue match of Lemma 3.26, so they carry no mass at any level and have $U(z) = 0$. For every direction with $v_3(q) = 1$ other than $\{1, 3\}$ the *growth rate of $M_n(z)$ is therefore strictly below φ. The bound is attained at $(1,12)$, where $t=1$, and at $(3,10)$, where $t=2$.*

Remark 3.37 (Two forms of the bound with no automaton in them). Cutting a closed path at its first return to the start state gives $\sum_n N(0,n)x^n = 1/(1-\Phi(x))$ with $\Phi(\varphi^{-1}) = \varphi^{-1}(1+U(z))$, so for a direction with $U(z) < \varphi^{-1}$

$$\sum_{n\geq 0}(M_n(z)+1)\varphi^{-n} = \frac{1}{\varphi^{-2}-\varphi^{-1}U(z)},$$

and $U(z) \leq \varphi^{-2}$ says exactly that this golden series is at most $\varphi^4 = 3\varphi+2$. Grouping by multiplier instead of by level: let $\mathcal{M}(z)$ be the set of $m \geq 1$ with $mz \in G_n$ for some n, and for $m \in \mathcal{M}(z)$ let $\ell(m)$ be the number of base-3 digits of $(z_1+z_2)m$. Since mz_1 and mz_2 are then binary with disjoint supports, $(z_1+z_2)m$ is their carry-free sum, so $mz \in G_n$ holds for $n \geq \ell(m)$ and for no smaller n. Hence $\sum_n M_n(z)\varphi^{-n} = \varphi^2\sum_m \varphi^{-\ell(m)}$, and Conjecture 3.41 reads

$$\sum_{m\in\mathcal{M}(z)} \varphi^{-\ell(m)} \;\leq\; \varphi$$

at every non-shift direction: a weighted count of the multipliers of z, with no carry automaton in it, in which the weight of a multiplier is φ to the minus its level.

Proposition 3.38 (The box is settled at every level)**.** *Among the* 13158 *primitive z with $z_1 \leq 120$ and $z_1 \leq z_2 \leq 240$, exactly* 6566 *have $3 \nmid z_1z_2$ and are settled outright by Theorem 3.28(1); of the other* 6592*, exactly* 218 *have a live automaton larger than the start state and the remaining* 6374 *have $M_n(z) = 0$ at every level, so they need nothing. Every one of the* 218 *has out-degree at most two with its branch states in a single class modulo* 3 *and all its carries inside $[-z_1/2,\, z_2/2]$, the largest live set having* 37 *states. Of the* 218*, case (2) settles* 206*, of which* 107 *have $k=1$, and the twelve left over are*

$$(1,9),\ (1,27),\ (1,81),\ (1,90),\ (4,117),\ (9,73),$$
$$(9,82),\ (9,235),\ (10,81),\ (13,108),\ (27,217),\ (27,226).$$

The first three are shift rays and fall to case (3); the other nine carry explicit Fibonacci certificates, of denominators 18*,* 40*,* 381*,* 18*,* 2013*,* 18*,* 40*,* 2013 *and* 34*, verified in exact integer arithmetic. So the golden ceiling holds on the whole box at every level, not only at the levels enumerated in Fact 3.40.*

The box is settled a second time by one criterion rather than twelve special cases. The golden potential of Definition 3.31 is finite, strictly positive and in $\mathbf{Q}(\sqrt{5})$ on all 218 *directions, and $U(z) \leq \varphi^{-2}$ on exactly* 214 *of them; the four failures are the shift rays $(1,3)$, $(1,9)$, $(1,27)$ and $(1,81)$, each with $U = \varphi^{-1}$ exactly, as Remark 3.33 requires. So Theorem 3.32 settles every non-shift direction of the box and Theorem 3.28(3) settles the four shift rays. Over the* 218*, U takes* 57 *distinct values: φ^{-1} on the four shift rays, then*

$\varphi^{-2} = 2 - \varphi$ at $(1,12)$, $(3,10)$ and $(4,9)$, then $(9\varphi - 14)/2$ at $(1,90)$, $(9,82)$ and $(10,81)$, and nothing in between φ^{-2} and φ^{-1}.

Proposition 3.39 (Ray mass is carried by the weight)**.** *Let z be primitive of weight $w = z_1 + z_2$ and let $D_n(w)$ count the K with $0 < K < 3^n$, K binary in base 3 and $w \mid K$. If $mz \in G_n$ then mz_1 and mz_2 are binary with disjoint support, so $mw = mz_1 + mz_2$ is binary and $mw < 3^n$, and the supports of mz_1 and mz_2 partition that of mw. Since $\gcd(z_1, w) = 1$ the map $m \mapsto mw$ is injective with $mz_1 = z_1(mw)/w$, so, writing* supp *for the set of positions carrying a nonzero base-3 digit,*

$$M_n(z) = \#\{K : \ 0 < K < 3^n, \ K \text{ and } z_1K/w \text{ binary}, \ w \mid K, \\ \operatorname{supp}(z_1K/w) \subseteq \operatorname{supp}(K)\} \ \le \ D_n(w).$$

Fact 3.40 (The golden ceiling on ray mass)**.** The claim is that $M_n(z) \le M_n(1,3) = F(n+1) - 1$ for every $z \in \mathbf{Z}^2_{\ge 1}$ and every n: the shift ray $(1,3)$ is the heaviest ray of the gasket at every level. It is *proved* for every z in the box $z_1 \le 120$, $z_1 \le z_2 \le 240$ and every n, by Proposition 3.38, and for every z in the six adversarial families named below, by Theorem 3.32; the enumerations reported next are an independent *verification* of the same range, and in general the claim is open. The 13158 coprime directions of the box are enumerated at every $n \le 40$, with $(1,3)$ the sole attainer at $n = 40$. Six adversarial families chosen to favour a breach are enumerated at every $n \le 45$: all binary base-3 pairs below 3^7 (4221 coprime), all no-adjacent-ones pairs below 3^7 (253), all $(1,t)$ with $t < 3000$ (2998), all consecutive pairs below 1500 (1499), all $(s, 3s-1)$ with $s < 1200$ (1199), and all $(s, 3s+1)$ with $s < 1200$ (1199). No breach. Those six overlap, the second family lying inside the first, so their 11369 coprime members are only 10862 distinct directions, of which 717 already lie in the box and 10145 are new. Of the 10145, 9498 carry no mass at any level, 608 fall under Theorem 3.28(2) and 3 under case (3), so exactly 36 rest on the enumeration alone. The criterion of Theorem 3.32 is stronger on that list than the enumeration is: of the 647 occupied directions outside the box, 644 satisfy $U(z) \le \varphi^{-2}$ and so are proved at every level, and the three that do not are the shift rays $(1,243)$, $(1,729)$ and $(1,2187)$, again with $U = \varphi^{-1}$. Together with Proposition 3.38 that is 865 occupied directions of which 858 are proved by the potential and the remaining seven, all shift rays, by Theorem 3.28(3); nothing rests on the enumeration alone once Theorem 3.32 is applied. Non-coprime z need no separate check: $M_n(er) \le M_n(r)$ for every $e \ge 1$. The shift rays $(1, 3^j)$ all grow at φ itself; the next rate down is the supergolden $1.4655\ldots$, the root of $x^3 = x^2 + 1$, attained at $(1,12)$, $(3,10)$, $(4,9)$ and their relatives. So the spectrum of ray growth is φ on the shift rays, then a gap, then $1.4655\ldots$

3.5. **What is not proved.**

Conjecture 3.41 (The golden partition bound)**.** *$U(a,b) \le \varphi^{-2}$ for every primitive (a,b) with $a, b \ge 1$ that is not a shift ray.*

What it would give. *Fact 3.40 as a theorem. By Theorem 3.32 the bound gives the golden ceiling at every level for every non-shift direction, and Theorem 3.28(3) already gives it on the shift rays. It is a genuine reduction of shape: one inequality for each direction in place of one for each pair* (z, n)*, and the quantity being bounded is a single algebraic number, computed by one exact linear solve on the live states and lying in* $\mathbf{Q}(\sqrt{5})$*.*

Evidence. *Over the box and the six adversarial families,* 865 *directions are occupied and* 858 *obey the bound; the seven that do not are exactly the shift rays* $(1, 3^j)$ *for* $1 \le j \le 7$*, each with* $U = \varphi^{-1}$ *exactly, as Remark 3.33 proves it must be. The bound is attained, at* φ^{-2}*, only on the supergolden directions* $(1, 12)$*,* $(3, 10)$ *and* $(4, 9)$*, and* U *takes no value in the open interval* $(\varphi^{-2}, \varphi^{-1})$ *anywhere in that range. Each check is exact in* $\mathbf{Q}(\sqrt{5})$*, and each solution is confirmed to satisfy the two inequalities of Theorem 3.32 rather than merely to solve the system.*

More than evidence is now available. Theorem 3.36 proves the conjecture outright on an infinite class cut out by two 3*-adic conditions, which holds at* 261 *of the* 360 *occupied directions of that census with* $v_3(q) = 1$*; and Theorem 3.35 proves it, direction by direction and without solving a linear system, wherever a sweep of the degree potential falls to* φ^{-2}*. On the same* 865 *occupied directions the degree potential is a super-solution at* 851*, of which* 37 *have a branch state with two branching successors and so lie outside Theorem 3.28(2); sweeping settles* 849 *of them, at sweep depth* 1 *for* 760*, depth* 3 *for* 48*, depth* 4 *for* 31*, depth* 5 *for* 7 *and depth* 6 *for* 3*, the depth being the least one that works. What is left is* 16 *directions: the seven shift rays, which Theorem 3.28(3) closes, and*

$$
\begin{gathered}
(1, 756),\ (1, 2196),\ (1, 2214),\ (1, 2268),\ (1, 2430), \\
(13, 1080),\ (27, 730),\ (28, 729),\ (40, 1053),
\end{gathered}
$$

at each of which the exact solve still gives $U \le \varphi^{-2}$ *but no certificate of this shape does. Over the wider lab census of* 36037 *directions the same two tools leave* 8 *shift rays and* 21 *named directions.*

Failure mode. *The bound is strictly stronger than the ceiling it implies, since it also pins the growth rate at or below the supergolden number: it is the rate gap of Fact 3.40 and the ceiling folded into one statement. So no argument that tolerates rate* φ *can prove it, and in particular Theorem 3.28(2) does not, since* $(1, 3)$ *satisfies the no-double-branching hypothesis and has* $U = \varphi^{-1}$*. Theorem 3.35 obeys that constraint in the only way available: its potential* π_0 *is a super-solution at* $(1, 3)$ *as well, and the gain comes not from the potential but from the sweep, which at* $(1, 3)$ *never falls below* φ^{-1} *because the start state is a successor of the state* c_0*. The open ground has narrowed. Theorem 3.36 closes* $v_3(q) = 1$ *except where* $27 \mid q/3 - p$*; there the spine of branch states above* c_0 *is long, the estimate* $\theta_i = 1 - \varphi^{-(i+2)}$ *rises to* 1*, and the missing input is a bound below* φ^{-1} *at the non-branch siblings along that spine. For* $v_3(q) \ge 2$ *the burst of Lemma 3.34(1) is the obstruction in its* *sharpest form, and it constrains every potential, not a chosen class of them. Each state at burst depth $j \le k-2$ has out-degree two with neither successor the start state, so any π satisfying the first inequality of Theorem 3.32 obeys $\pi(c) \ge \varphi^{-1}\sum_{c'}\pi(c')$ there, and iterating down the burst gives*

$$\varphi^{-2} \;\ge\; \pi(c_0) \;\ge\; \varphi^{-(k-1)}\sum_m \pi(q_1 m),$$

the sum running over the 2^{k-1} distinct burst-floor states $q_1 m$ with $m < 3^k$ binary in base 3 and $m \equiv 1 \pmod 3$, all of them of valuation 0. So π averages at most $\varphi^{k-3}2^{-(k-1)}$ across the burst floor. When $2p \le q$ the state p is live with the start state as its only successor, forcing $\pi(p) \ge \varphi^{-1}$ although $v_3(p) = 0$ as well; the ratio of the two demands is $\varphi^2(2/\varphi)^{k-1}$, which grows without bound. A valid potential must therefore separate states of equal valuation by an exponentially large factor: no π constant on the level sets of v_3 can work once $k \ge 2$, and neither can π_0 of Theorem 3.35, which is constant on the out-degree classes. What could grade finely enough is exactly what is open. The identity above is checked in exact $\mathbf{Q}(\sqrt{5})$ arithmetic against $\pi = u$, together with $u(p) = \varphi^{-1}$ whenever $2p \le q$, on all 865 occupied directions of the census.

Conjecture 3.42 (The residual is $o(3^n)$)**.** *$R(n)/3^n \to 0$, that is, $E(n) = 2\cdot 3^n + o(3^n)$ by Proposition 3.10. This is the weak form: Conjecture 3.43 below asserts the sharp rate $R(n) = O(\varphi^{2n})$ and implies this statement, since $\varphi^2 < 3$, but the weak form is all that the second-moment application needs.*

Evidence. *Theorem 3.9 settles the dominant part outright: the shift rays contribute $(2.1982\ldots + o(1))\varphi^{2n}$ to E, and $\varphi^2 = 2.618\ldots < 3$. Fact 3.14 measures R to level 17: $R(n)/3^n$ falls at every level from $n = 8$ onward, and $R(n)/\varphi^{2n}$ peaks at 3.2378 at $n = 12$ and then falls at five consecutive levels to $2.7724831\ldots$. Theorem 3.24 closes the weight-four orbit outright, 194096 of $R(13) = 863848$, by the same Fibonacci mechanism as Theorem 3.9.*

Failure mode. *What Theorem 3.9 does not touch is the non-shift rays, which carry between 0.16 and 0.35 of R at every level measured. Bounding them by multiplier pair means bounding $\sum \#\{z : sz, tz \in G_n\}$ over the non-shift active pairs, a set that grows: at $n = 4,\ldots,13$ those pairs number 10, 30, 106, 332, 1010, 2642, 7564, 20934, 57858, 154410, and the largest multiplier occurring is exactly $\lfloor 3^n/8\rfloor$ at each of those levels, as Lemma 3.20 requires. A per-pair bound $\lambda^\flat(s,t) \le 2$ buys nothing there unless it comes with a constant decaying in (s,t) fast enough to beat that count, and Fact 3.18 is a finite check at $\max(s,t) \le 52$, not a uniform statement.*

Definition 2.12 moves the sum off that index set. In the witness coordinate the constant is not missing but golden: $R(n) = \sum_z P_n(z)$ with $P_n(z) \le M_n(z)^2 - M_n(z)$, and Fact 3.40 caps every $M_n(z)$ by $F(n+1)-1 < \varphi^{n+1}/\sqrt{5}$. What is still owed is summability, stated next.

Conjecture 3.43 (The residual is golden)**.** *$R(n) = O(\varphi^{2n})$; equivalently, in the witness coordinate of Definition 2.12,*

$$\sum_{z \in \mathbf{Z}^2_{\geq 1}} P_n(z) \;=\; O(\varphi^{2n}).$$

Since $\varphi^2 = 2.618\ldots < 3$ this implies Conjecture 3.42, and by Proposition 3.10 it would put $E(n) = 2 \cdot 3^n + O(\varphi^{2n})$.

Evidence. *Fact 3.14: $R(n)/\varphi^{2n}$ rises to 3.2378 at $n = 12$ and then falls at every level to $2.7724831\ldots$ at $n = 17$, with the level ratio at $2.5073119\ldots$, below φ^2. Theorem 3.24 proves the bound for the weight-four orbit with the explicit constant 1.6945, and Theorem 3.9 proves it for the shift family with 2.803. Fact 3.40 supplies the per-witness cap that the per-pair route of Remark 3.45 never had.*

Failure mode. *The cap of Fact 3.40 is uniform in z but the sum is not: witnesses of weight w exist up to $w = (3^n - 1)/2$ by Lemma 3.20, so the index set still grows with n, and $M_n(z) \leq F(n+1) - 1$ alone gives only $R(n) \leq \varphi^{2n} \cdot \#\{z : M_n(z) \geq 2\}/5$, which is far too lossy: the crude majorant $\sum_z M_n(z)(M_n(z) - 1)$ already grows at 2.907 per level at $n = 13$, against 2.573 for R itself, because it drops the coprimality of (s,t). The missing lemma is a decay of $M_n(z)$ in the weight strong enough to make $\sum_w R_w(n)$ converge after the weight-four orbit is removed; along the geometric subsequence $w = 3^k + 1$, which is not the list of the heaviest layers but is the one whose members are comparable, the measured masses at $n = 13$ are 120148, 62436, 35292, 16424, 9324, 7600, 4228 at $w = 4$, 10, 28, 82, 244, 730, 2188, each roughly half its predecessor. Proposition 3.39 supplies the first half of such a lemma outright, $M_n(z) \leq D_n(w)$, and it is the wrong half by itself. The rate of $D_n(w)$ is 2, not φ: at $n = 24$ the counts $D_{24}(w)$ at $w = 4$, 10, 28, 82 are 4196351, 1683971, 613817, 228519, against the ceiling $F(25) - 1 = 75024$, so for fixed w the weight enters the bound through its constant only. What is still owed is a bound whose rate falls with w, or a summation that keeps the support condition dropped in passing from the displayed set to $D_n(w)$. Granting Conjecture 3.41, and with it the ceiling for every z, would leave that debt exactly where it stands: the ceiling bounds each summand of $R(n) = \sum_z P_n(z)$ and says nothing about how many summands there are, so the statement still to prove is precisely $\sum_{w>4} R_w(n) = O(\varphi^{2n})$, the whole sum after Theorem 3.24 has removed the weight-four orbit. Coprimality has to survive that summation: the majorant $\sum_z M_n(z)(M_n(z) - 1)$, which keeps the ceiling and drops only the coprimality of (s,t), already runs at 2.907 per level, above φ^2, so no route through the second moment of the mass alone can reach the conjecture.*

Conjecture 3.44 (The gap is universal)**.** *For every coprime pair (s,t) with $s < t$, $\lambda(s,t) \in \{3\} \cup [0,2]$; that is, the open interval $(2,3)$ contains no Perron root of any $\mathcal{A}(s,t)$, and $\lambda(s,t) = 3$ only for shift pairs.*

Evidence. *All* 829 *coprime pairs with* $\max(s,t) \le 52$ *obey it, each certified by exact integer arithmetic rather than by a numerical eigenvalue. The values cluster:* 3*, then* 2*, then* $1.6956\ldots$ *and below, with a clean gap of width* $0.30\ldots$ *under* 2 *as well as the full width of* $(2,3)$ *above it. The state counts stay small, at most* 45 *reachable and at most* 33 *live over the whole range, so the automata do not appear to be growing in a way that would eventually fill the interval.*

Failure mode. *A pair with a large multiplier could have a much larger reachable state set and a Perron root anywhere in* $(2,3)$*; nothing in the finite check bounds the state count for general* (s,t)*. A proof would need a uniform argument, for instance a subinvariant vector* $v > 0$ *with* $\mathcal{A}v \le 2v$ *constructed for every non-shift pair at once.*

Remark 3.45 (Open direction)*.* Fact 3.17 bounds pairs, not triples or larger families, and a bound on pairs alone gives only a crude bound on the number of occupied gasket rays. Extending the gap from pairs to k-tuples of multipliers is the natural next question and is not addressed here. Neither is the one thing Conjecture 3.42 would need from it: a bound with a constant explicit in (s,t), since the pairs active at level n already number about 2.77^n and any bound of the shape $C(s,t)\,2^n$ is useless without control of $\sum C(s,t)$. That is a reason to change coordinate rather than to sharpen the pair bound: Conjecture 3.43 asks the same question of the witness, where Fact 3.40 already supplies the constant and only the summation over weights is open.

4. Proofs

Proof of Lemma 3.1. Write $x = \sum_{i<n} 3^i (d_i, \phi(d_i))$, and likewise for x' with digits d'_j. Bilinearity gives

$$\det(x,x') = \sum_{i,j<n} 3^{i+j}\, w(d_i, d'_j), \qquad w(u,v) := u\,\phi(v) - \phi(u)\,v.$$

The form w is antisymmetric, $w(v,u) = -w(u,v)$, and $w(u,u) = 0$.

Collect the coefficient of 3^m:

$$C_m = \sum_{\substack{i+j=m\\ 0\le i,j<n}} w(d_i, d'_j).$$

Fix $m < k$. Every index occurring in C_m is at most $m < k$, and d and d' agree below k, so $d'_j = d_j$ for all such j and $C_m = \sum_{i+j=m} w(d_i, d_j)$. Pair the term (i,j) with the term (j,i): the two cancel by antisymmetry, and a term with $i = j$ vanishes on its own. Hence $C_m = 0$ for every $m < k$, and $v_3(\det(x,x')) \ge k$.

Now take $m = k$. The terms are indexed by $i = 0, \ldots, k$ with $j = k - i$. For $1 \le i \le k-1$ both indices are strictly below k, so as above those terms cancel in pairs and contribute nothing. What remains is $i \in \{0, k\}$:

$$C_k = w(d_0, d'_k) + w(d_k, d'_0).$$

If $k = 0$ the two displayed terms are the single term $w(d_0, d_0')$, and the computation below still applies verbatim. If $k \geq 1$ then $d_0' = d_0$, so, using antisymmetry,

$$C_k = w(d_0, d_k') - w(d_0, d_k) = \phi(d_0)\big(d_k - d_k'\big) - d_0\big(\phi(d_k) - \phi(d_k')\big).$$

It remains to evaluate C_k modulo 3. Every permutation of $\mathbf{F}_3$ is affine: the affine maps $x \mapsto \alpha x + \beta$ with $\alpha \in \{1, 2\}$ and $\beta \in \{0, 1, 2\}$ are six distinct permutations, and $\mathbf{F}_3$ has only six permutations, so they are all of them. Writing $\phi(x) = \alpha x + \beta$ and noting $\beta = \phi(0)$,

$$\begin{aligned} C_k &\equiv (\alpha d_0 + \beta)(d_k - d_k') - d_0\, \alpha\, (d_k - d_k') \\ &= \beta\, (d_k - d_k') \equiv \phi(0)\, (d_k - d_k') \pmod 3. \end{aligned}$$

This is the stated congruence. If $\phi(0) \neq 0$ then both factors of $\phi(0)(d_k - d_k')$ are nonzero in the field $\mathbf{F}_3$, because $d_k \neq d_k'$, so $C_k \not\equiv 0 \pmod 3$ and $v_3(\det(x, x')) = k$ exactly. A number of finite 3-adic valuation is nonzero. □

Proof of Theorem 3.2. Suppose $\phi(0) \neq 0$. Then no digit of F_ϕ is $(0, 0)$, so no level-n point is the origin for $n \geq 1$. Distinct digit strings give distinct points, because the first coordinate $\sum_i d_i 3^i$ with $d_i \in \{0, 1, 2\}$ is a base-3 representation and so recovers the string. By Lemma 3.1, for any two distinct digit strings the cross determinant is nonzero, so the two points are not collinear with the origin. Hence the 3^n points sit on 3^n distinct rays, every occupied ray has mass 1, and F_ϕ is diagonal.

Count the fibre rays. A level-n point has second coordinate 0 exactly when $\phi(d_i) = 0$ for every i, that is, when every digit equals $c := \phi^{-1}(0)$; since $\phi(0) \neq 0$ we have $c \neq 0$, and the single string $d_i = c$ for all i gives one point, which is nonzero because its first coordinate is $c(3^n - 1)/2 > 0$. So the fibre ray $(1, 0)$ is occupied, by exactly one point. Symmetrically the first coordinate vanishes exactly when every digit is 0, giving the single point $(0, \phi(0)(3^n - 1)/2)$, so the fibre ray $(0, 1)$ is occupied by exactly one point. No point lies on both, since it would be the origin. Therefore exactly two of the 3^n occupied rays are fibre rays and

$$Z_{F_\phi}(n) = 3^n - 2 \qquad (n \geq 1),$$

which is the claim.

Conversely suppose $\phi(0) = 0$. There are two such permutations. For the identity, $F_{\mathrm{id}} = \{(0, 0), (1, 1), (2, 2)\}$ and at level 1 the points $(1, 1)$ and $(2, 2)$ satisfy $1 \cdot 2 - 1 \cdot 2 = 0$, so they are collinear with the origin and F_{id} is not diagonal. For $\phi(j) = 2j$, that is $F_\phi = \{(0, 0), (1, 2), (2, 1)\}$, level 1 is fine but level 2 is not: the digit string $(d_0, d_1) = (1, 0)$ gives $(1, 2)$ and the string $(0, 1)$ gives $(3, 6)$, and $1 \cdot 6 - 2 \cdot 3 = 0$. So F_ϕ is not diagonal either. This proves the equivalence, and the four qualifying permutations are exactly those with $\beta = \phi(0) \neq 0$, namely $x \mapsto x + 1$, $x \mapsto x + 2$, $x \mapsto 2x + 1$ (the swap of 0 and 1) and $x \mapsto 2x + 2$ (the swap of 0 and 2). □

Proof of Proposition 3.5. The level-n gasket has 3^n points, one per digit string, and distinct strings give distinct points because the pair of coordinates determines each digit. A point lies on the fibre ray $(1,0)$ or is the origin exactly when its second coordinate vanishes, that is when every digit lies in $\{(0,0),(1,0)\}$: that is 2^n strings. Symmetrically 2^n strings give second-coordinate points with first coordinate 0. The two families meet in the single all-$(0,0)$ string, the origin. So the number of points that are nonzero and off both fibres is

$$3^n - \bigl(2^n + 2^n - 1\bigr) = 3^n - 2^{n+1} + 1,$$

and this count is precisely the sum of the masses of the non-fibre rays, since every such point lies on exactly one primitive ray. □

Proof of Proposition 3.7. A point $x \in G_n$ is a digit string $d_0, \dots, d_{n-1} \in G$, and x lies on the ray through (a,b) exactly when $b\,x_1 - a\,x_2 = 0$. Run $\mathcal{C}(a,b)$ on the string. After i steps the state is $(b\,u_1 - a\,u_2)/3^i$, where u is the point built from the first i digits; admissibility is exactly the statement that this is an integer, and the run returns to 0 after n steps precisely when $b\,x_1 - a\,x_2 = 0$. Closed paths of length n at the start state are therefore in bijection with the level-n gasket points on the ray, the origin included, so their number is $M_n(a,b)+1$. Trimming to live states changes nothing, as in Proposition 3.16. For the bound, the increment $b\,d_1 - a\,d_2$ takes only the values 0, b, $-a$ as d runs over G, so $c \mapsto (c+\varepsilon)/3$ carries $[-a/2,\, b/2]$ into itself: the largest image is $(b/2+b)/3 = b/2$ and the smallest is $(-a/2-a)/3 = -a/2$. The start state 0 lies in that interval. □

Proof of Theorem 3.8(1), (3) and (4). By Proposition 3.7 it suffices to exhibit the live automaton in each case; then $(T^n)_{00}$ satisfies the linear recurrence whose polynomial is the characteristic polynomial of T, by Cayley–Hamilton, and matching as many initial values as the degree pins the sequence for every n.

For $(3,1)$ the increment is $c \mapsto c + d_1 - 3d_2$. From 0 the digit $(0,0)$ returns to 0, the digit $(1,0)$ gives 1, not divisible by 3, and the digit $(0,1)$ gives -3, hence the state -1. From -1 the digit $(1,0)$ gives 0 and the other two give -1 and -4, neither divisible by 3. So the live states are $\{0,-1\}$ with $T = \left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$, whose powers are the Fibonacci matrix, $(T^n)_{00} = F(n+1)$.

For $(1,12)$ the increment is $c \mapsto c + 12d_1 - d_2$. From 0 the digit $(0,0)$ returns to 0 and the digit $(1,0)$ gives 12, hence the state 4; from 4 only $(0,1)$ survives, giving 3, hence 1; from 1 only $(0,1)$ survives, giving 0. The live set is $\{0,4,1\}$, a loop at 0 together with a 3-cycle through it, so $\det(\lambda I - T) = \lambda^3 - \lambda^2 - 1$ and $(T^n)_{00}$ obeys $a(m) = a(m-1) + a(m-3)$ with $(T^0)_{00}, (T^1)_{00}, (T^2)_{00} = 1, 1, 1$.

For $(7,3)$ the increment is $c \mapsto c + 3d_1 - 7d_2$. From 0 the digit $(0,0)$ returns and $(1,0)$ gives 3, hence 1; from 1 only $(0,1)$ survives, giving -6, hence -2; from -2 only $(0,1)$ survives, giving -9, hence -3; from -3 the digit $(1,0)$ gives 0 and the digit $(0,0)$ gives -3, hence -1, from which no digit is admissible. The live set is $\{0,1,-2,-3\}$, a loop at 0 together with a 4-cycle, so $\det(\lambda I - T) = \lambda^4 - \lambda^3 - 1$ and $(T^n)_{00}$ obeys $c(m) = c(m-1) + c(m-4)$ with $(T^3)_{00}, \dots, (T^6)_{00} = 1, 2, 3, 4$. $\square$

Proof of Theorem 3.8(2). Multiplication by 3^j shifts base-3 digits, so at position i the point $z\,(3^j, 1) = (3^j z,\, z)$ has digit pair $(z_{i-j},\, z_i)$, with z_{i-j} read as 0 for $i < j$. That pair lies in $G = \{(0,0),(1,0),(0,1)\}$ exactly when both entries lie in $\{0,1\}$ and they are not both 1. Also $3^j z < 3^n$ exactly when $z < 3^{n-j}$. So the multipliers z with $z\,(3^j, 1) \in G_n$ are exactly the strings $z_0, \dots, z_{n-j-1}$ over $\{0,1\}$ with no two ones at distance j.

Sort the positions $0, \dots, n-j-1$ by residue modulo j. The constraint links two positions only when they are congruent modulo j and adjacent inside their class, so the string factors as j independent binary strings with no two adjacent ones, of lengths $m_r = \#\{i \in [0, n-j) : i \equiv r \bmod j\}$. Binary strings of length m with no two adjacent ones number $F(m+2)$, so the count is $\prod_{r<j} F(m_r+2)$, and removing $z = 0$ gives the stated mass. Writing $n-j = qj+\rho$ with $0 \le \rho < j$ gives $m_r = q+1$ for $r < \rho$ and $m_r = q$ otherwise, which is the block form. The mirror identity $M_n(1, 3^j) = M_n(3^j, 1)$ holds because G is stable under swapping the two coordinates. $\square$

Proof of Theorem 3.9. First, $F(k) \le 2\varphi^{k-3}$ for every $k \ge 2$, by induction on two steps. The bases are $F(2) = 1 \le 2\varphi^{-1} = 1.236\ldots$ and $F(3) = 2 = 2\varphi^0$, the latter with equality. For $k \ge 4$, assuming the bound at $k-1$ and $k-2$,

$$\begin{aligned} F(k) = F(k-1) + F(k-2) &\le 2\varphi^{k-4} + 2\varphi^{k-5} \\ &= 2\varphi^{k-5}(\varphi+1) = 2\varphi^{k-5}\varphi^2 = 2\varphi^{k-3}, \end{aligned}$$

using $\varphi^2 = \varphi + 1$. That is $F(k) \le (2/\varphi^3)\varphi^k$, with equality exactly at $k = 3$. Applying it to each factor of Theorem 3.8(2), and using $\sum_{r<j}(m_r + 2) = (n-j) + 2j = n + j$,

$$\prod_{r<j} F(m_r+2) \;\le\; (2/\varphi^3)^j\,\varphi^{n+j} = (2/\varphi^2)^j\,\varphi^n = (3-\sqrt{5})^j\,\varphi^n,$$

since $1/\varphi^2 = 2-\varphi$ and $2(2-\varphi) = 3-\sqrt{5}$. Subtracting 1 makes the inequality strict. Squaring, summing over $j \ge 1$ and doubling for the mirror rays,

$$\mathrm{Sh}(n) < 2\varphi^{2n} \sum_{j\ge1} (3-\sqrt{5})^{2j} = 2\varphi^{2n}\frac{14-6\sqrt{5}}{6\sqrt{5}-13} = \frac{4+12\sqrt{5}}{11}\,\varphi^{2n},$$

the geometric series converging because $(3-\sqrt{5})^2 = 14-6\sqrt{5} = 0.5835\ldots <$ 1.

For the limit, fix j and let $n \to \infty$. Then $q = \lfloor (n-j)/j \rfloor \to \infty$, and $F(k) = \varphi^k/\sqrt{5} + O(\varphi^{-k})$ gives

$$F(q+3)^\rho F(q+2)^{j-\rho} = \frac{\varphi^{\rho(q+3)+(j-\rho)(q+2)}}{5^{j/2}}(1+o(1)) = \frac{\varphi^{n+j}}{5^{j/2}}(1+o(1)),$$

because $\rho(q+3)+(j-\rho)(q+2) = qj+\rho+2j = n+j$. Hence $M_n(3^j,1)^2/\varphi^{2n} \to (\varphi^2/5)^j$ for each fixed j. The summands are dominated uniformly in n by the summable $(3-\sqrt{5})^{2j}$ just proved, so dominated convergence applies term by term and

$$\frac{\mathrm{Sh}(n)}{\varphi^{2n}} \longrightarrow 2\sum_{j\geq 1}\Big(\frac{\varphi^2}{5}\Big)^j = \frac{2\varphi^2}{5-\varphi^2} = \frac{13+5\sqrt{5}}{11}. \qquad \square$$

Proof of Proposition 3.10. Multiplication by 3^j shifts digits, so for $z \in \mathbf{Z}^2$ the condition $3^j z \in G_n$ says exactly $z \in G_{n-j}$, which already forces $z \in G_n$. The nonzero non-fibre points of G_{n-j} number $3^{\,n-j} - 2^{\,n-j+1} + 1$ by Proposition 3.5. Every ordered off-diagonal collinear pair with multiplier ratio a power of 3 is $(z, 3^j z)$ or $(3^j z, z)$ for a unique $j \geq 1$ and a unique such z, so summing over j and doubling gives $\Sigma(n)$. The closed form follows from $\sum_{j=1}^{n-1} 3^{\,n-j} = (3^n-3)/2$ and $\sum_{j=1}^{n-1} 2^{\,n-j+1} = 2^{n+1}-4$. $\square$

Proof of Lemma 2.10. At the start state the first carry is $\mathbf{a} = (0,0)$, so the admissibility test on the first track reads $d \bmod 3 \in G$, which for $d \in \{0,1,2\}^2$ says $d \in G$; and the successor first carry is $\lfloor d/3 \rfloor = (0,0)$. By induction the first carry is $(0,0)$ at every reachable state and every admissible digit lies in G, so $\mathcal{B}(1,t)$ and $\mathcal{A}(1,t)$ have the same reachable states and the same edges. $\square$

Proof of Proposition 3.16. A point $z \in G_n$ is the same thing as a digit string $d_0, \ldots, d_{n-1} \in G$, and distinct strings give distinct points. Run $\mathcal{A}(s,t)$ on that string. By construction the state after i steps is the pair of carries left by multiplying z by s and by t, and the digit d_i is admissible exactly when the base-3 digit at position i of sz and of tz both lie in G. If the path is back at the start state after n steps then both carries vanish, so sz and tz have all digits in G and nothing above position $n-1$, that is $sz, tz \in G_n$; conversely if $sz, tz \in G_n$ then every step is admissible and the carries after n steps are zero, so the path returns. Closed paths of length n at the start state are therefore in bijection with the points counted. No closed path through the start state visits a state that is not live, so the count is unchanged by the trimming and equals $(T^n)_{00}$.

Every live state lies on a closed path through the start state, hence is in the same strongly connected component as the start state, so the live digraph is strongly connected. The digit $(0,0)$ is admissible at the start state and returns to it, so the digraph carries a loop. A nonnegative matrix whose digraph is strongly connected and carries a loop is primitive, and for a primitive matrix $T^n = \lambda^n(vw^{\mathsf{T}} + o(1))$ with $v, w > 0$, so $(T^n)_{00}^{1/n} \to \lambda$ [5, Ch. 1]. $\square$

Proof of Proposition 3.19. Take $s = 1$, $t = 4$ in Definition 2.7. From the start state $A = (0,0;0,0)$ the digit $(0,0)$ produces output digits $(0,0)$ and $(0,0)$ with no carry, returning to A; the digit $(1,0)$ produces $1 \cdot (1,0) = (1,0)$, admissible with no carry, and $4 \cdot (1,0) = (4,0) \equiv (1,0) \bmod 3$ with carry $(1,0)$, so the successor is $B = (0,0;1,0)$; symmetrically the digit $(0,1)$ leads to $C = (0,0;0,1)$. From B, the digit $(0,0)$ gives second-track value $(1,0)$, admissible, with successor carry $(0,0)$, so $B \to A$; the digits $(1,0)$ and $(0,1)$ give second-track values $(5,0) \equiv (2,0)$ and $(1,1)$ respectively, neither in G, so they are inadmissible. Symmetrically $C \to A$ only. No further states are reachable, so the reachable set is $\{A, B, C\}$ with the transition matrix

$$T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Its trace is 1, the sum of its principal 2×2 minors is $(-1) + (-1) + 0 = -2$, and its determinant is 0 because the last two rows agree; hence $\det(\lambda I - T) = \lambda^3 - \lambda^2 - 2\lambda = \lambda(\lambda - 2)(\lambda + 1)$ and the spectral radius is 2.

For the count, let G_m be the number of paths of length m starting at A. Such a path either begins with the loop $A \to A$, leaving a path of length $m-1$ from A, or begins with $A \to B$ or $A \to C$, in which case the next step is forced back to A and leaves a path of length $m-2$ from A. Hence $G_m = G_{m-1} + 2G_{m-2}$ for $m \geq 2$, with $G_0 = 1$ and $G_1 = 3$. Solving the recurrence with roots 2 and -1 gives $G_m = \bigl(2^{m+2} - (-1)^m\bigr)/3$, and $G_{15} = (2^{17} + 1)/3 = 43691$. The same split applies to paths that return to A, since after $A \to B \to A$ or $A \to C \to A$ the walk is again at A: writing R_m for their number, $R_m = R_{m-1} + 2R_{m-2}$ for $m \geq 2$ with $R_0 = R_1 = 1$, so $R_m = \bigl(2^{m+1} + (-1)^m\bigr)/3$ and $R_{15} = (2^{16} - 1)/3 = 21845$. By Proposition 3.16 it is R_m, not G_m, that counts the z with $z, 4z \in G_m$. □

Proof of Lemma 3.20. A pair (X, Y) lies in G_n exactly when X and Y have base-3 digits in $\{0, 1\}$ and disjoint digit support; equivalently, when X, Y and $X + Y$ all have base-3 digits in $\{0, 1\}$, since disjointness is exactly the absence of carries in $X + Y$. Suppose $mz \in G_n$ with $m \geq 1$ and $z_1, z_2 \geq 1$, and put $w = z_1 + z_2 \geq 2$. If $w = 2$ then $z = (1,1)$ and $mz_1 = mz_2 = m$, whose support meets itself unless $m = 0$. If $w = 3$ then z is $(1,2)$ or $(2,1)$; the coordinate with multiplier 1 forces m itself to be binary, so its lowest nonzero digit is 1, and then $2m$ carries the digit 2 in that position, so $2m$ is not binary. Both cases are impossible, so $w \geq 4$.

For the height, tz_1 and tz_2 lie in disjoint digit positions and are binary, so $tz_1 + tz_2 = tw$ is binary and less than 3^n; the largest binary base-3 number below 3^n is $(3^n - 1)/2$, whence $tw \leq (3^n - 1)/2$ and $t \leq (3^n - 1)/8$. The same argument applies to s. □

Proof of Proposition 3.21. Let $t = \max(s, t) > (3^n - 1)/10$. By the proof of Lemma 3.20 any witness has $tw \leq (3^n - 1)/2$, so $w < 5$, and $w \geq 4$ gives $w = 4$. The witnesses of weight 4 are $(1,3)$, $(2,2)$ and $(3,1)$, and $(2,2)$ is excluded because $2m$ would have to be disjoint from itself. Exchanging coordinates is a symmetry of G and hence of G_n, and it exchanges $(1,3)$ with $(3,1)$, so the two witnesses are admissible together. Each admissible witness contributes the unordered point pair $\{sz, tz\}$, that is two ordered pairs, so the pair carries 4 or 0. □

Proof of Proposition 3.22. A $\mathcal{B}$-edge on the digit (d_x, d_y) from $(\mathbf{a}; \mathbf{b})$ requires $\big((sd_x{+}a_1) \bmod 3, (sd_y{+}a_2) \bmod 3\big) \in G$ and the same for t, and its successor acts on the x-carries (a_1, b_1) using d_x alone and on the y-carries (a_2, b_2) using d_y alone. So the underlying transition is the product of two $\mathcal{C}$-steps, and only the admissibility couples them. Since $G = \{(0,0),(1,0),(0,1)\}$, membership says each of the four residues lies in $\{0,1\}$ and neither the two s-residues nor the two t-residues are both 1. The first requirement is already imposed inside $\mathcal{C}$. Writing the two forbidden events as $\{u_x = u_y = 1\}$ and $\{v_x = v_y = 1\}$, inclusion and exclusion on the product $S \otimes S$ removes $U \otimes U$ and $V \otimes V$ and restores their intersection $W \otimes W$. The carry bound is immediate: $a \le s$ and $b \le t$ are preserved by $a \mapsto \lfloor (sd + a)/3 \rfloor$ since $d \le 2$. □

Proof of Proposition 3.23. By the proof of Lemma 3.20, $t(z_1 + z_2) \le (3^n - 1)/2$ for every z counted on the left, so the constraint $z_1 + z_2 \le W_n(t)$ is vacuous and the two sets are equal. The right-hand set is contained in a triangle of $O(W_n(t)^2)$ lattice points, and each membership test reads n base-3 digits. □

Proof of Theorem 3.24. By the proof of Proposition 3.21 the weight-four witnesses are exactly $(1,3)$ and $(3,1)$, which are exchanged by the coordinate swap, so $R_4(n)$ is twice the count for $z = (1,3)$. There m is admissible exactly when m and $3m$ are binary with disjoint support, that is when m is binary with no two adjacent ones and $3m < 3^n$; the binary strings of length $n-1$ with no two adjacent ones number $F(n+1)$, and discarding $m = 0$ gives $\#F_n = F(n{+}1){-}1$. The pairs counted are the ordered coprime pairs from F_n whose ratio is not a power of 3, so $R_4(n) < 2(\#F_n)^2$. From $F(k) = (\varphi^k - \psi^k)/\sqrt{5}$ with $|\psi| < 1 < \sqrt{5}$ we get $F(n+1) - 1 < \varphi^{n+1}/\sqrt{5}$, hence $R_4(n) < 2\varphi^{2n+2}/5 = (2\varphi^2/5)\varphi^{2n} < 1.0473\,\varphi^{2n}$.

For the scaling, $3v \in G_n$ if and only if $v \in G_{n-1}$, since multiplying by 3 shifts every base-3 digit up one place. So $m(3z) \in G_n$ if and only if $mz \in G_{n-1}$, giving $M_n(3z) = M_{n-1}(z)$ and $P_n(3z) = P_{n-1}(z)$. It remains to see that the witnesses of weight $3w$ are exactly 3 times those of weight w, that is, that a witness z of weight divisible by 3 has both coordinates divisible by 3. Suppose $3 \mid z_1 + z_2$ but $3 \nmid z_1$, so also $3 \nmid z_2$. Then for any $m \ge 1$, $v_3(mz_1) = v_3(m) = v_3(mz_2)$, so mz_1 and mz_2 have their lowest nonzero base-3 digit in the same position; being binary, that digit is 1 in both, so their supports meet and $mz \notin G_n$. Hence such a z is a witness for no m at all, and summing $P_n(3z) = P_{n-1}(z)$ over the witnesses of weight w gives $R_{3w}(n) = R_w(n-1)$. Iterating and summing the geometric series, $\sum_{j\ge 0} R_4(n-j) < (2\varphi^2/5)\varphi^{2n} \sum_{j \ge 0} \varphi^{-2j} = (2\varphi^3/5)\varphi^{2n} < 1.6945\,\varphi^{2n}$, using $\varphi^2/(\varphi^2 - 1) = \varphi$. □

Proof of Lemma 3.25. (1) A prime dividing both a and b divides their gcd, which is 1, so 3 cannot divide both. If it divides a then $a + b$ is congruent to b, which it does not divide, so it does not divide $a + b$; the case $3 \mid b$ is symmetric. At most one of the three is a multiple of 3.

(2) If $3 \nmid ab(a+b)$ the three increments are pairwise incongruent modulo 3, so each state admits exactly one increment and the automaton is deterministic; from the start state the admissible increment is 0, and the unique path of every length is the all-$(0,0)$ one. If instead $3 \nmid ab$ but $3 \mid a+b$, then $b \equiv -a \not\equiv 0$, so 0 is the only increment congruent to 0: the start state has out-degree one with itself as successor, and again the all-$(0,0)$ path is the only closed one at 0. In both cases $M_n(a,b) = 1-1 = 0$ by Proposition 3.7.

(3) By (1) and (2) exactly one of a, b is divisible by 3; call it q. The congruent pair of increments is $\{0,-a\}$ or $\{0,b\}$ according as q is a or b, and in each case the two differ by $\pm q$; the third increment is not divisible by 3, so it sits in a different class. A state c admits ε exactly when $\varepsilon \equiv -c$, so out-degree two occurs exactly on $3 \mid c$ and out-degree one on the class of the third increment; the remaining class has out-degree zero. The successors of a branch state are $(c+\varepsilon_1)/3$ and $(c+\varepsilon_2)/3$, which differ by $(\varepsilon_1-\varepsilon_2)/3 = \pm q/3$, and $3 \nmid q/3$ exactly when $v_3(q) = 1$. □

Proof of Theorem 3.28. (1) is Lemma 3.25(2).

(2) Write $N(c,n)$ for the number of paths of length n from the state c to the start state 0, and $G(n) = \max_c N(c,n)$ over the live states, a finite set by Proposition 3.7. Then $N(c,0) = [c=0]$, so $G(0) = 1 = F(1)$; and the successors of a state are distinct, so at most one of them is 0 and $G(1) \le 1 = F(2)$. Let $n \ge 2$. If c has out-degree at most one then $N(c,n) \le G(n-1)$. If c branches, its two successors are c' and c'' and by hypothesis one of them, say c'', is not a branch state; then $N(c'',n-1)$ is 0 if c'' has out-degree zero and otherwise equals $N(c''',n-2) \le G(n-2)$ for the unique successor c''' of c''. So $N(c,n) \le G(n-1)+G(n-2)$ in every case, and $G(n) \le G(n-1)+G(n-2)$. Induction gives $G(n) \le F(n+1)$, and $M_n(a,b) = N(0,n)-1 \le F(n+1)-1$ by Proposition 3.7.

(3) For $n \le j$ the mass is 0 and the claim is clear. For $n > j$, Theorem 3.8(2) gives $M_n(1,3^j)+1 = \prod_{r<j} F(m_r+2)$ with $m_r \ge 0$ and $\sum_r m_r = n-j$. The addition formula $F(p+q+3) = F(p+2)F(q+2)+F(p+1)F(q+1)$, valid for $p,q \ge 0$, gives

$$F(p+2)F(q+2) = F(p+q+3) - F(p+1)F(q+1) \le F(p+q+3) - 1,$$

since $F(p+1)F(q+1) \ge 1$. Merging the j factors one pair at a time replaces indices $p+2$ and $q+2$ by $p+q+3$, so the index total drops by exactly one per merge and after $j-1$ merges the bound is $F\big(\sum_r m_r + 2j - (j-1)\big) = F(n+1)$. For $j \ge 2$ at least one merge occurs, so the product is at most $F(n+1)-1$ and the mass at most $F(n+1)-2$.

(4) We show $N(c,m) \le \alpha(c)F(m+1) + \beta(c)F(m)$ for every live c and every $m \ge 0$, by induction on m. At $m=0$ the right side is $\alpha(c)$ and the left is $[c=0]$, so the base case is $\alpha(0) \ge 1$ and $\alpha(c) \ge 0$. For the step,

$N(c,m+1) = \sum_{c'} N(c',m)$ over successors with multiplicity, so

$$N(c,m+1) \le \Big(\sum_{c'} \alpha(c')\Big)F(m+1) + \Big(\sum_{c'} \beta(c')\Big)F(m)$$
$$\le (\alpha(c)+\beta(c))F(m+1) + \alpha(c)F(m),$$

using $F(m+1) \ge 0$ and $F(m) \ge 0$, and the right side is $\alpha(c)F(m+2) + \beta(c)F(m+1)$. Taking $c = 0$ gives $N(0,n) \le F(n+1)$, hence the ceiling. $\square$

Proof of Theorem 3.32. Write $G_M(c) = \sum_{m \le M} g(c,m)\varphi^{-m}$, so $G_M(0) = 1$ for every M, and $g(c,m) = \sum_{c'} g(c',m-1)$ for $c \ne 0$ and $m \ge 1$, whence $G_M(c) = \varphi^{-1}\sum_{c'} G_{M-1}(c')$. We claim $G_M(c) \le \pi(c)/\pi(0)$ for every live c and every M. At $M = 0$ the left side is $[c = 0]$, and the claim reads $1 \le 1$ at the start state and $0 \le \pi(c)/\pi(0)$ elsewhere. For the step nothing changes at $c = 0$, while at $c \ne 0$

$$G_M(c) = \varphi^{-1}\sum_{c'} G_{M-1}(c') \le \frac{\varphi^{-1}}{\pi(0)}\sum_{c'} \pi(c') \le \frac{\varphi^{-1}\varphi\,\pi(c)}{\pi(0)} = \frac{\pi(c)}{\pi(0)}.$$

Letting M grow gives $u(c) \le \pi(c)/\pi(0) < \infty$, and summing over the successors of the start state other than itself gives $U(a,b) \le \varphi^{-2}$, hence $S := \sum_{j\ge 2} f_j\varphi^{-j} = \varphi^{-1}U(a,b) \le \varphi^{-3}$.

Next two envelopes: $F(m) \le \varphi^{m-1}$ for every $m \ge 0$ and $F(m) \ge \varphi^{m-2}$ for every $m \ge 1$. Both hold at $m = 0$, 1, 2 by inspection: $F(0) = 0 < \varphi^{-1}$, and $F(1) = F(2) = 1$ against the bounds $\varphi^{-1} \le 1 \le \varphi^0$ at $m = 1$ and $\varphi^0 \le 1 \le \varphi$ at $m = 2$. Both are then inherited along $F(m) = F(m-1)+F(m-2)$ from $\varphi^{m-1} = \varphi^{m-2} + \varphi^{m-3}$.

Write $N(c,n)$ for the number of paths of length n from c to the start state, so $M_n(a,b) = N(0,n)-1$ by Proposition 3.7. Cutting a closed path of length $n \ge 1$ at its first return to the start state gives $N(0,n) = \sum_{j=1}^{n} f_jN(0,n-j)$. We show $N(0,n) \le F(n+1)$ by induction. At $n = 0$ and $n = 1$ both counts are $1 = F(1) = F(2)$, using $f_1 = 1$: the increments are distinct because $a,b \ge 1$, and the only one carrying the start state to itself is 0. For $n \ge 2$,

$$N(0,n) \le F(n) + \sum_{j=2}^{n} f_jF(n+1-j) \le F(n) + \varphi^n S$$
$$\le F(n) + \varphi^{n-3} \le F(n) + F(n-1) = F(n+1),$$

the first step by the induction hypothesis and $f_1 = 1$, the second by $F(n+1-j) \le \varphi^{n-j}$, the third by $S \le \varphi^{-3}$ and the fourth by $F(n-1) \ge \varphi^{n-3}$.

For the converse, suppose $U(a,b) \le \varphi^{-2}$. Every live c is reached from the start state by a path, and after its last visit to the start state that path runs from some successor $c' \ne 0$ to c in ℓ steps without meeting the start state; concatenating it with the first-hitting paths out of c shows $u(c') \ge \varphi^{-\ell}u(c)$, so u is finite everywhere. It is positive because a live state reaches the start state, $u(0) = 1$, and $\sum_{c'} u(c') = \varphi\, u(c)$ at every $c \ne 0$ by the recursion above; the remaining hypothesis is $U(a,b) \le \varphi^{-2}$ itself. $\square$

Proof of Lemma 3.26. If $M_n(z) > 0$ there is an $m \geq 1$ with $mz \in G_n$, so mp and mq are both binary in base 3. Write $m = 3^s m'$ with $3 \nmid m'$. Then $mp = 3^s(m'p)$ and $mq = 3^{s+k}(m'q_1)$, and multiplying by a power of 3 only shifts a base-3 expansion, so $m'p$ and $m'q_1$ are binary as well. Neither is divisible by 3, since $3 \nmid m'$, $3 \nmid p$ and $3 \nmid q_1$; a binary base-3 number with last digit not 0 has last digit 1, so $m'p \equiv m'q_1 \equiv 1 \pmod 3$. Cancelling m', which is invertible modulo 3, gives $p \equiv q_1$. □

Proof of Lemma 3.34. The map $c \mapsto -c$ carries $\mathcal{C}(b,a)$ to $\mathcal{C}(a,b)$, because negating the increment set $\{0, a, -b\}$ gives $\{0, b, -a\}$ and the start state is fixed; so u, the f_j and U are unchanged by swapping the coordinates and we may assume $3 \mid b$, that is $q = b$ and $p = a$. Then the two increments congruent to 0 are 0 and b, so by Lemma 3.25(3) a state c has out-degree two exactly when $3 \mid c$, with successors $c/3$ and $c/3 + b/3$; out-degree one exactly when $c \equiv a$, with successor $(c-a)/3$; and out-degree zero when $c \equiv -a$. The start state therefore has exactly one successor other than itself, namely $c_0 = b/3$, and the states with the start state as a successor are a and $-b$.

(1) Write $c_0 = 3^{k-1}q_1$ and let $0 \leq j \leq k-1$. Every state reached from c_0 in j steps has the form $3^{k-1-j}q_1\bigl(1+\sum_{i\in T} 3^i\bigr)$ with $T \subseteq \{1, \ldots, j\}$: this holds at $j = 0$, and a state of that shape with $j \leq k-2$ is divisible by 3, so its two successors are obtained by dividing by 3 and optionally adding $b/3 = 3^{k-1}q_1$, which is the shape at $j+1$. All those states are positive, hence different from the start state, so $g(c_0, j) = 0$ for $j \leq k-1$ and $f_j = g(c_0, j-1) = 0$ for $2 \leq j \leq k$.

(2) This is the count in Remark 3.33.

(3) A first return of length 3 is $0 \to c_0 \to c_2 \to 0$ with $c_2 \neq 0$, so $c_2 \in \{a, -b\}$ and c_2 is a successor of c_0, that is $3c_2 = b/3 + \varepsilon$ with $\varepsilon \in \{0, b, -a\}$. Taking $c_2 = a$: $\varepsilon = 0$ gives $b = 9a$, so $a = 1$ and $b = 9$; $\varepsilon = b$ gives $4b = 9a$, so $a = 4$ and $b = 9$; $\varepsilon = -a$ gives $b = 12a$, so $a = 1$ and $b = 12$. Taking $c_2 = -b$: $\varepsilon = 0$ and $\varepsilon = b$ ask $-3b$ to equal $b/3$ or $4b/3$, impossible for $b \geq 1$, and $\varepsilon = -a$ gives $3a = 10b$, so $b = 3$ and $a = 10$. Each of the four is realised, with the required congruence $\varepsilon \equiv -c_0$ holding and c_2 inside the band of Proposition 3.7, and each admits exactly one such path, so $f_3 = 1$ there. At $\{1, 3\}$ the state $c_0 = 1$ has the start state as its only successor, so $f_3 = 0$.

(4) By Definition 3.31, $\sum_{j\geq 2} f_j \varphi^{-j} = \varphi^{-1}U(z)$, and $f_2 = 0$ by (2), so $U(z) = \varphi^{-2}\sum_{j\geq 3} f_j \varphi^{3-j}$. The terms are nonnegative, so a sum of at most 1 forces $f_3 \leq 1$; with $f_3 = 0$ it forces $f_4 \leq \varphi$, that is $f_4 \leq 1$; and with $f_3 = 0$ and $f_4 = 1$ it leaves $\sum_{j\geq 5} f_j \varphi^{5-j} \leq 1$. □

Proof of Theorem 3.35. Suppose no branch state of the live automaton has both successors of out-degree two. A live state of out-degree one has a single successor c' with $\pi_0(c') \leq 1 = \varphi \cdot \varphi^{-1} = \varphi\, \pi_0(c)$. A live state $c \neq 0$ of out-degree two has successors c_1, c_2 of which at most one has out-degree two, the start state counting as such a successor since its out-degree is two; so

$\pi_0(c_1) + \pi_0(c_2) \le 1 + \varphi^{-1} = \varphi = \varphi\,\pi_0(c)$. That is the first inequality of Theorem 3.32.

Now assume only that π_0 satisfies it. Write L for the operator $(L\pi)(0) = 1$, $(L\pi)(c) = \varphi^{-1}\sum_{c'}\pi(c')$ at live $c \ne 0$, so $\pi_d = L^d\pi_0$. The hypothesis is $L\pi_0 \le \pi_0$, and L is monotone because its coefficients are nonnegative, so $\pi_{d+1} \le \pi_d$ by induction. In the notation of the proof of Theorem 3.32, $G_M = L^M G_0$ and $G_0 \le \pi_0$, since G_0 is the indicator of the start state and $\pi_0(0) = 1$; monotonicity gives $G_M \le L^M\pi_0 = \pi_M \le \pi_d$ for $M \ge d$, and letting M grow gives $u \le \pi_d$ for every d. Summing over the successors of the start state other than itself gives $U(a,b) \le \sum_{c' \ne 0}\pi_d(c')$, and if that is at most φ^{-2} then Theorem 3.32 applies with $\pi = u$. □

Proof of Theorem 3.36. As in the proof of Lemma 3.34 take $3 \mid b$, so $b = q = 3q_1$ and $a = p$, out-degree two on $3 \mid c$, out-degree one on $c \equiv a$ and out-degree zero on $c \equiv -a$. By Lemma 3.26 $q_1 \equiv a \pmod 3$, so $3 \mid q_1 - a$ and $t \ge 1$; and $t = \infty$ means $q_1 = a$, whence $1 = \gcd(a,b) = \gcd(a,3a) = a$ and $z = \{1,3\}$, which is excluded. By Lemma 3.25(3) with $v_3(q) = 1$ the two successors of a branch state differ by q_1, which is not divisible by 3, so at most one of them is divisible by 3 and hence has out-degree two; Theorem 3.35 therefore applies and $u \le \pi_0$, where $\pi_0 \le 1$ everywhere.

The start state has the single other successor $c_0 = q_1$, which is not divisible by 3 and is congruent to a, so it has the single successor $x = (q_1 - a)/3 \ne 0$ and $U(z) = u(c_0) = \varphi^{-1}u(x)$.

If $t = 1$ then $3 \nmid x$, so $\pi_0(x) = \varphi^{-1}$ and $U(z) \le \varphi^{-2} = \varphi^{-1}(1 - \varphi^{-2})$.

If $t \ge 2$, put $y_j = x/3^j$ for $0 \le j \le t-1$, so $v_3(y_j) = t - 1 - j$ and $y_j \ne 0$. For $0 \le j \le t-2$ the state y_j is divisible by 3 and its successors are y_{j+1} and $y_{j+1} + q_1$. At $j = t-2$ the state y_{t-1} is not divisible by 3, so with r its class modulo 3 the two successor classes are r and $r + a$ with $r \not\equiv 0$; whether $r \equiv a$ or $r \equiv -a$, one of $\{r, r+a\}$ is the class $-a$ of out-degree zero, so that successor is not live and contributes $u = 0$; the other contributes at most 1, being either the start state or a live state with $u \le \pi_0 \le 1$ or a state that is not live with $u = 0$. Hence $u(y_{t-2}) \le \varphi^{-1}\cdot 1 = \varphi^{-1}$. For $j \le t-3$ the state y_{j+1} is divisible by 3, so $y_{j+1} + q_1 \equiv q_1 \equiv a$ has out-degree at most one, and it is not the start state because $3 \nmid q_1$; so either it is live, where $u(y_{j+1} + q_1) \le \pi_0(y_{j+1} + q_1) = \varphi^{-1}$, or it is not, where $u(y_{j+1} + q_1) = 0$. Hence $u(y_j) \le \varphi^{-1}\big(u(y_{j+1}) + \varphi^{-1}\big)$ for $j \le t-3$. Setting $\theta_0 = \varphi^{-1}$ and $\theta_i = \varphi^{-1}(\theta_{i-1} + \varphi^{-1})$ gives $u(y_{t-2-i}) \le \theta_i$, and $\theta_i = 1 - \varphi^{-(i+2)}$ solves that recursion since $\varphi^{-1}(1 + \varphi^{-1}) = 1$. Taking $i = t-2$ gives $u(x) = u(y_0) \le 1 - \varphi^{-t}$ and $U(z) \le \varphi^{-1}(1 - \varphi^{-t})$.

The two cases combine as $U(z) \le \varphi^{-1}(1 - \varphi^{-\max(t,2)}) < \varphi^{-1}$, and $1 - \varphi^{-\max(t,2)} \le \varphi^{-1} = 1 - \varphi^{-2}$ exactly when $\max(t,2) = 2$. Finally $U(z) < \varphi^{-1}$ makes $\Phi(\varphi^{-1}) = \varphi^{-1}(1 + U(z)) < 1$, so $\sum_n N(0,n)x^n = 1/(1 - \Phi(x))$ has radius of convergence larger than φ^{-1} and $N(0,n)$ grows at a rate strictly below φ. □

Proof of Proposition 3.39. Let $mz \in G_n$. Then mz_1 and mz_2 are binary in base 3, below 3^n, with disjoint supports, so the base-3 expansion of $K = mz_1 + mz_2 = mw$ has no carries: it is binary, it is below 3^n, and its support is the disjoint union of the two. Distinct m give distinct K, and $z_1K/w = mz_1$ recovers the first coordinate, so the displayed set contains the image of $m \mapsto mw$. Conversely a K in that set has $w \mid K$, and putting $m = K/w$ makes $mz_1 = z_1K/w$ and $mz_2 = K - mz_1$ binary with disjoint supports inside that of K, so $mz \in G_n$. The two sets therefore correspond, and dropping the support condition gives $D_n(w)$. □

5. Reproducibility

Two scripts ship with the paper. Both are plain `python3` with no dependencies outside the standard library, take no arguments, read nothing, and use only relative paths. Run them from the paper's directory.

`python3 scripts/verify.py` runs in about 105 seconds and recomputes every number in Section 3, with the single flagged exception noted below. It covers, in order:

- all 84 digit designs at levels $1 \le n \le 6$, confirming the ten survivors and the four permutation designs of Fact 3.4, their codes $84, 98, 140, 266$ under the indexing of Definition 2.3, the fibre counts, and that the cell set named by 148 is not a permutation design;
- the cross coefficient of Lemma 3.1 in all 18 cases (d_0, d_k, d'_k) for each of the six permutations, confirming both that it is nonzero exactly when $\phi(0) \neq 0$ and that it equals $\phi(0)(d_k - d'_k)$ modulo 3;
- $Z_{F_\phi}(n) = 3^n - 2$ for the four diagonal designs and $1 \le n \le 8$, $Z_{F_\phi}(0) = 0$, and the two explicit collinear witnesses of Theorem 3.2;
- Theorem 3.8: rays $(3, 1)$, $(1, 12)$ and $(7, 3)$ for $n \le 30$, the thirteen shift rays $(3^j, 1)$ for $1 \le j \le 13$ and $n \le 30$ against the Fibonacci product, and the bounded-carry automaton against direct enumeration of all 3^n points for six rays and $1 \le n \le 10$, and against the split enumeration for the same six rays and $1 \le n \le 14$; then the three live automata of the proof, their state counts 2, 3, 4, their exact characteristic polynomials, and the annihilator residuals of each claimed recurrence;
- Theorem 3.9: the run-length characterisation of the shift multipliers by direct enumeration for $n \le 11$, the block form for $n \le 60$, the integer lemma $F(k) \le 2F(k-1)$, the per-j bound and the family bound in exact $\mathbf{Z}[\sqrt{5}]$ arithmetic for $n \le 160$, and a two-sided rational window placing $\mathrm{Sh}(n)/\varphi^{2n}$ in (2.198212, 2.198214) for $100 \le n \le 160$;
- Proposition 3.10 for $n \le 20$, and Fact 3.14, both series by literal enumeration;
- Fact 3.15: the level-9 census, its 2656 active ordered multiplier pairs, the 482 with a witness off the gasket and the 2540 pairs they miss,

$\mathcal{A}(s,t)$ against brute force on all 1328 unordered pairs, and $\mathcal{B}(s,t)$ against brute force on the 103 of height at most 52;

- the perfect-square levels of Remark 3.12 over $1 \le n \le 30$, and the two cases that make $x^4 - x^3 - 1$ irreducible;
- Proposition 3.5 at every level $1 \le n \le 12$, and Fact 3.13 by enumerating all 531441 level-12 points, including the ten heaviest rays and the mirror symmetry;
- Fact 3.17: all 829 coprime unordered pairs with $\max(s,t) \le 52$, each trimmed to its live states and given an exact integer characteristic polynomial with a nonnegative-shift certificate, plus a bisection in exact rational arithmetic and two-sided rational windows pinning $1.6956207695598\ldots$ with its 25 attainers and $1.6769719158912\ldots$ with its four, the largest reachable and live state counts 45 and 33, the trimming trap of Remark 2.8 at $(1,16)$, the return counts of Proposition 3.16 against direct enumeration for $n \le 8$, and the explicit data of Proposition 3.19 including the recurrence residuals through $m = 15$;
- Proposition 3.22: the tensor identity against the return counts of $\mathcal{B}(s,t)$ on all 473 coprime pairs below 40 at every level to 9, and the four carry-against-reachable state counts;
- Proposition 3.23: the box construction against $\mathcal{B}(s,t)$ on all 812 coprime pairs with $s < 30$, $t < 60$ at $n = 9$, and against Proposition 3.22 at five large-multiplier pairs at $n = 9$ and $n = 12$;
- Lemma 3.20, Proposition 3.21 and Theorem 3.24: $R(n)$ and its weight layers by literal enumeration for $4 \le n \le 13$, the scaling law on all 1869 layers of weight divisible by 3, the absence of any weight below 4, the sharpness of the height bound at $\lfloor 3^n/8 \rfloor$ for $4 \le n \le 13$, the exactly-four contribution of all 30028 pairs above $(3^n - 1)/10$ for $6 \le n \le 13$, and $R_4(n)$ against the no-adjacent-ones count with $\#F_n = F(n+1) - 1$ for $4 \le n \le 12$;
- Fact 3.40: $M_n(1,3) = F(n+1) - 1$ for $n \le 40$, the ceiling over all 13158 coprime directions in the box at every $n \le 40$ with $(1,3)$ the sole attainer at $n = 40$, and the ceiling over the six adversarial families at every $n \le 45$, family by family with their coprime counts 4221, 253, 2998, 1499, 1199, 1199;
- Lemma 3.25, Theorem 3.28 and Proposition 3.38: the ray automaton rebuilt from the increments $\{0, b, -a\}$ and matched against the gasket-digit automaton on all 947 coprime pairs below 40 for $n \le 20$; then over the box, the carry interval $[-z_1/2, z_2/2]$, the out-degree ceiling of two, the single branch class, the counts 13158, 6566, 6592, 218, 107, 206 and the twelve exceptions by name; the nine Fibonacci certificates checked in exact integer arithmetic; then Definition 3.31 and Theorem 3.32, the golden potential solved in exact $\mathbf{Q}(\sqrt{5})$ arithmetic on all 218 occupied directions of the box and on all 647 occupied outside it, each solution confirmed against both inequalities

of the theorem, the counts 214 and 644 passing, the seven shift rays over the criterion at $U = \varphi^{-1}$ exactly, and the maximum φ^{-2} with its three attainers; the addition formula $F(p+q+3) = F(p+2)F(q+2) + F(p+1)F(q+1)$ for $p, q < 60$ and the shift-ray product against $F(n+1)$ for $j \le 13$, $n \le 45$; and the counterexample of Remark 3.30 with its eight failing directions;

- Proposition 3.39: $M_n(z) \le D_n(w)$ on all 829 coprime directions with $z_1 \le 30$, $z_1 \le z_2 \le 60$ at $n = 12$, and $D_{24}(w)$ at $w = 4$, 10, 28, 82 against $F(25) - 1$;
- the extension of Fact 3.14 to $n = 13$ and $n = 14$ by the direction-grouping pass. Levels 15, 16 and 17 run the identical pass on a machine-integer array sort rather than a list of Python integers, which is a change of container and not of method; they are outside the shipped script only because 3^{17} points do not fit its time budget;
- Lemma 3.26, Lemma 3.34, Theorem 3.35 and Theorem 3.36 over the union of the box and the six families, 23303 directions of which 11691 have $3 \mid z_1z_2$, 7103 match the residue condition and 865 are occupied: no occupied direction fails the residue match, no occupied direction has a first return of length between 2 and $v_3(q)$, the burst identity of Conjecture 3.41 and the value $u(p) = \varphi^{-1}$ hold at all 865, the degree potential is a super-solution at 851 of the 865 and at 37 directions outside Theorem 3.28(2), its sweep settles 849 with the least-depth split 760, 48, 31, 7, 3 at depths 1, 3, 4, 5, 6 and leaves the seven shift rays and the nine directions named in Conjecture 3.41, and the bound of Theorem 3.36 holds at all 360 occupied directions with $v_3(q) = 1$, covers 261 of them and is attained exactly at $(1, 12)$ and $(3, 10)$; the classification of f_2 and f_3 rechecked by direct enumeration on all occupied directions with $z_1 < 130$ and $z_2 < 800$;
- Fact 3.18: the same 829 pairs read through $\mathcal{B}$, its largest live state set 167, its agreement with $\mathcal{A}$ on every pair with $s = 1$ as Lemma 2.10 requires, the same three, twenty and empty lists, then its own ceiling below 2 pinned in a two-sided rational window with its four attainers, the split $19+25 = 44$ of pairs that reach or beat the gasket-digit ceiling, certified by the strict rational on one side and by exact divisibility of the characteristic polynomial by $x^3 - x^2 - 2$ on the other, and the closed-path witness at $n = 60$.

Every assertion names the quantity, the value obtained, and the value expected, and the script stops at the first mismatch. A successful run ends with `all green` and exit status 0.

`python3 scripts/figure.py` runs in well under a second and writes `figures/rays.svg`, the two-panel picture used outside the paper; the version inside the paper, Fig. 1, is drawn in TEX.

`tectonic paper.tex` rebuilds `paper.pdf`.

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