PARITY CARPETS CORRELATE BY GCD: FOUR EXACT LAWS FOR A SQUARE-WAVE STACK

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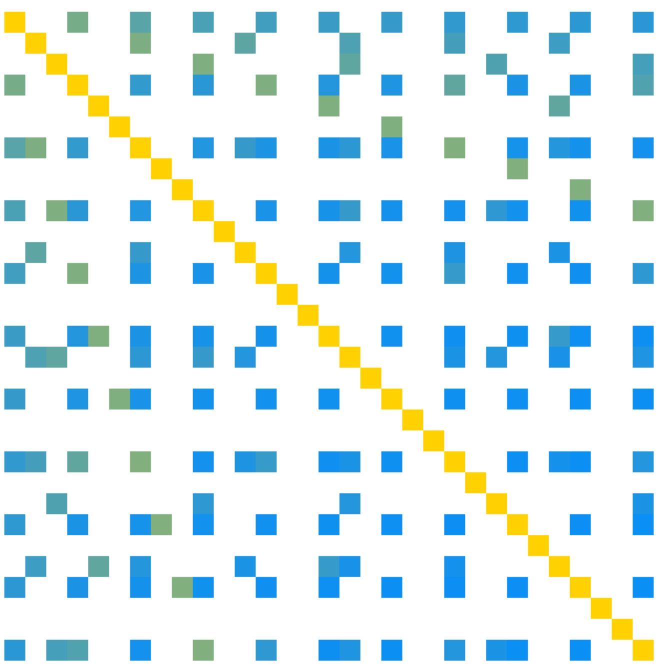

Abstract. Cut a square into an odd number of equal strips each way and ink every cell whose two strip numbers are both odd. Two such pictures at different scales overlap by an amount fixed by the greatest common divisor of the scales, and that amount is exact. We prove the master integral behind it, derive the exact correlation law for each of the four fields the parity rule generates, and show that all four vanish precisely when the scales are coprime. So an odd number above one is prime exactly when its picture is uncorrelated with every picture before it: a portrait of coprimality, not a primality test. Two ray laws are proved alongside, and two boundaries: the exactness fails in a hexagonal section, and the stack's brightness carries no Möbius information.

1. Introduction

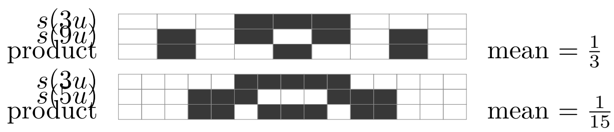

Figure 1. Two parity waves and their product; a cell is dark where the wave is -1. The product's mean is exactly $\gcd(m,n)^2/(mn)$: one third for $(3,9)$, one fifteenth for the coprime $(3,5)$.

Here is a picture anyone can draw. Fix an odd number n, cut the unit square into n columns and n rows numbered from zero, and shade a cell when both its numbers are odd. Draw a second sheet at a different odd n and hold the two up to the light: how much do they agree? The answer is one exact number. With $g = \gcd(m,n)$ for the two scales, the correlation of the sheets has g in its numerator and is *exactly zero* when $g = 1$ – not small, not zero to five decimals, zero.

1.1. **The master integral, and what is classical about it.** Everything rests on one integral. With $s(x) = (-1)^{\lfloor x \rfloor}$ the square wave of period 2, for all positive integers m, n with $g = \gcd(m,n)$,

$$\int_0^1 s(mu)\, s(nu)\, du = \frac{g^2}{mn} \quad \text{if } m/g \text{ and } n/g \text{ are both odd,} \qquad 0 \text{ otherwise.}$$

Figure 1 is that statement drawn twice.

This identity has a famous sibling. With $((x))$ the sawtooth $x - \lfloor x \rfloor - \frac{1}{2}$ (and $((x)) = 0$ at integers),

$$\int_0^1 ((mx))\,((nx))\, dx = \frac{\gcd(m,n)^2}{12\, mn},$$

which is the two-fold Franel integral, standard in the literature of Dedekind sums [1], the base case of the family whose four-fold generalisation was conjectured by McIntosh and proved recently [3], and the computational heart of the Franel-Landau reformulation of the Riemann hypothesis. Our master integral is the same computation with the sawtooth's full harmonic comb replaced by the square wave's odd-only comb, by the same Parseval argument. That substitution is exactly where the hypothesis "m/g and n/g both odd" comes from, and why the constant is 1 rather than 1/12. A referee should read Theorem 3.1 as a gcd-sum evaluation of a routine kind, not as a new theorem; the classical identity is re-verified alongside ours in `scripts/verify.py`.

The increment this paper offers is narrower and, we think, more honest to state plainly. First, exactness: the author's own earlier and unpublished working notes on stacked designs already recorded that the cross-layer correlation is about zero at gcd $= 1$ and that it tracks $\gcd^2/(mn)$; what is proved here is that it is *exactly* zero, with closed forms for all four fields and with the coprimality criterion as an identity rather than an observation. Second, the ray laws: the two-adic criterion that a line through the origin of slope q/p carries signal if and only if p and q are both odd. Third, the negative results, which we regard as the most useful part: the failure in the hexagonal section, and the Möbius refutation. The prime corollary itself claims no novelty at all – as an algorithm it is trial division with a picture attached.

1.2. **What this paper proves.** The parity of m/g and n/g is not decoration: at $(m, n) = (1, 2)$ the integral is 0 rather than 1/2, and the unqualified formula fails on 532 of the 820 unordered pairs with $m, n \leq 40$. From the master integral, four correlation laws drop out, one for each black-and-white field the parity rule generates (Theorem 3.3), and all four vanish at exactly the same places (Theorem 3.4). Since an odd composite always has an odd prime factor smaller than itself, the sheet of a composite always echoes a smaller sheet and the sheet of a prime never does (Corollary 3.5). The pictures see coprimality; primality is what coprimality against everything smaller happens to mean.

The paper also proves two exact ray laws for the stacked picture (Theorem 3.8, Proposition 3.9), gives the stack's variance exactly, as a finite rational sum (Proposition 3.12), and then marks two boundaries, because a result is only as valuable as the map of where it stops. The plane rule that makes all of this work has a nonzero pair term in its harmonic expansion; the analogous rule in space has none, and with it goes the mechanism (Proposition 3.19). And the brightness of the stacked picture, tempting as it looks, contains no Möbius information whatsoever (Proposition 3.21); no Riemann-hypothesis criterion can be read off it.

2. Definitions

Definition 2.1. $s(x) = (-1)^{\lfloor x \rfloor}$ is the square wave of period 2, and $\chi_n(u) = \mathbf{1}[\lfloor nu \rfloor \text{ odd}] = \frac{1}{2}(1 - s(nu))$ is the odd-strip indicator at scale n. All integrals are over $u \in [0,1)$ or $(u,v) \in [0,1)^2$ with Lebesgue measure, and $\langle \cdot \rangle$ denotes the mean.

Definition 2.2. For odd $n \geq 1$ the parity rule generates four $\{0,1\}$-valued fields on the unit square:

$$\mathrm{T}_n(u,v) = \chi_n(u), \qquad \mathrm{C}_n(u,v) = \chi_n(u)\chi_n(v),$$

$$\mathrm{N}_n(u,v) = \big(1-\chi_n(u)\big)\big(1-\chi_n(v)\big), \qquad \mathrm{V}_n(u,v) = \tfrac{1}{2}\big(1 + s(nu)s(nv)\big).$$

We call them *tree*, *carpet*, *net* and *void*. The tree depends on one coordinate; the carpet marks cells whose two strip numbers are both odd; the net is its complementary product; the void marks cells whose two strip numbers have the same parity.

The sheet of Section 1 is the carpet. Printing it the other way round – inking every cell *except* those with both indices odd – gives the field $1 - \mathrm{C}_n$, and correlation is invariant under $X \mapsto 1 - X$ applied to both arguments, so every statement below about C holds verbatim for that picture too. Of the four, C, N and V are the three non-constant fields one can build from the pair $(\chi_n(u), \chi_n(v))$ that respect the symmetry $u \leftrightarrow v$, counted up to complementation: as functions of two bits they are AND, NOR and XNOR, and the remaining three symmetric non-constant functions NAND, OR and XOR are their complements. The tree T is not symmetric at all – it reads one coordinate and ignores the other – and is kept because it is the simplest law of the family.

Definition 2.3. For fields X, Y on the unit square, $\mathrm{Cov}(X,Y) = \langle XY \rangle - \langle X \rangle \langle Y \rangle$ and $r(X,Y) = \mathrm{Cov}(X,Y)/\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}$. The *stack* of the first L odd scales is $\frac{1}{L}\sum_{n \in S} \mathrm{C}_n$ with $S = \{1, 3, \ldots, 2L-1\}$.

Because each field is $\{0,1\}$-valued, $r = 0$ is more than decorrelation: at a uniformly random point of the square the two values are then independent Bernoulli random variables. The fields themselves are deterministic; the independence is of the two readings taken at one random point, and we never claim more.

3. Results

Theorem 3.1 (master integral)**.** *Let $m, n \geq 1$ be integers, $g = \gcd(m,n)$, $m' = m/g$, $n' = n/g$. Then*

$$\int_0^1 s(mu)\, s(nu)\, du = \begin{cases} \dfrac{g^2}{mn}, & m' \text{ and } n' \text{ both odd}, \\ 0, & \text{otherwise}. \end{cases}$$

Corollary 3.2. *For odd m, n the integral is $g^2/(mn)$; in particular $\langle s(nu) \rangle = 1/n$ and $\langle \chi_n \rangle = \frac{n-1}{2n}$ for odd n.*

The right-hand side is rational, and this is worth a sentence: the square wave's Fourier normalisation contributes $(4/\pi)^2$ and the odd Basel sum $\sum_{k \text{ odd}} k^{-2} = \pi^2/8$ eats it exactly. Every observable in this paper is rational for the same reason. Reporting π as a feature of these pictures would be numerology.

Theorem 3.3 (four correlation laws)**.** *Let $m, n \geq 3$ be odd, $d = \gcd(m, n)$. Then*

$$\mathrm{Cov}(\mathrm{T}_m, \mathrm{T}_n) = \frac{d^2 - 1}{4mn}, \quad r(\mathrm{T}_m, \mathrm{T}_n) = \frac{d^2 - 1}{\sqrt{(m^2 - 1)(n^2 - 1)}},$$

$$\mathrm{Cov}(\mathrm{C}_m, \mathrm{C}_n) = \frac{(d^2 - 1)\left[2(m-1)(n-1) + d^2 - 1\right]}{16m^2n^2},$$

$$r(\mathrm{C}_m, \mathrm{C}_n) = \frac{(d^2 - 1)\left[2(m-1)(n-1) + d^2 - 1\right]}{(m-1)(n-1)\sqrt{(3m-1)(m+1)(3n-1)(n+1)}},$$

$$\mathrm{Cov}(\mathrm{N}_m, \mathrm{N}_n) = \frac{(d^2 - 1)\left[2(m+1)(n+1) + d^2 - 1\right]}{16m^2n^2},$$

$$r(\mathrm{N}_m, \mathrm{N}_n) = \frac{(d^2 - 1)\left[2(m+1)(n+1) + d^2 - 1\right]}{(m+1)(n+1)\sqrt{(3m+1)(m-1)(3n+1)(n-1)}},$$

$$\mathrm{Cov}(\mathrm{V}_m, \mathrm{V}_n) = \frac{d^4 - 1}{4m^2n^2}, \qquad r(\mathrm{V}_m, \mathrm{V}_n) = \frac{d^4 - 1}{\sqrt{(m^4 - 1)(n^4 - 1)}}.$$

The four formulas are different functions and must not be substituted for one another. At $(m, n) = (3, 9)$ they read 0.316227766 for the tree, 0.262950294 for the net, 0.219264505 for the carpet and 0.110431526 for the void: one zero set, four different sizes. The tree law is the simplest and the most quotable, but the picture a reader draws from Section 1 is the carpet, and the carpet's law is the third one.

Theorem 3.4 (one zero set, and strict positivity off it)**.** *For odd $m, n \geq 3$, all four covariances of Theorem 3.3 are* 0 *if* $\gcd(m, n) = 1$ *and all four are strictly positive if* $\gcd(m, n) > 1$. *In particular the four fields share one zero set, and* $\mathrm{Cov} = 0 \iff \gcd(m, n) = 1$ *in each family.*

Corollary 3.5 (the coprimality detector)**.** *Let $n \geq 3$ be odd and let $\mathcal{F}$ be any one of the four families. Then n is prime if and only if* $\mathrm{Cov}(\mathcal{F}_m, \mathcal{F}_n) = 0$ *for every odd m with $3 \leq m < n$.*

This is independent of any window: it holds for the infinite family of odd scales, not merely inside some finite stack. A finite stack can only distort it. In a stack of the odd scales up to 55, for instance, the scales with no echo at all are $19, 23, 29, 31, 37, 41, 43, 47, 53$, which are the primes above $55/3$; that list is an artefact of the window, since 17 is dropped only because $51 = 3 \cdot 17$ happens to be in the stack. The window statement should never be quoted as the theorem; Corollary 3.5 is the theorem.

Fact 3.6. Score each odd n in $3 \le n \le 199$ by $\max\{r(\mathrm{C}_m, \mathrm{C}_n) : m \text{ odd}, 3 \le m < n\}$, the score of $n = 3$ being 0 over an empty set. Then all 45 primes in the range score exactly 0 and all 54 composites score strictly positive, with no overlap. The narrowest composite is $n = 169 = 13^2$ at 0.0517383422; the same n scores 0.0766964989 in the tree family and 0.0059170562 in the void family. Checked by `scripts/verify.py`.

Remark 3.7. Corollary 3.5 is not a primality test worth running. Deciding it for one n means testing $\gcd(m, n) = 1$ for every smaller odd m, which is trial division with a picture attached; it detects primes, it does not explain them. Its interest is that a purely geometric question – do these two drawings agree more than chance? – has an arithmetic answer with no error term, and that the primes are exactly the drawings that are new. The primes so produced are of course A000040.

Theorem 3.8 (the slope-one ray law)**.** *Let $S = \{1, 3, \ldots, 2L-1\}$ and, for real c, let*

$$Q_L(c) = \frac{1}{L} \sum_{n \in S} \int_0^1 s(nu)\, s(nu + nc)\, du.$$

If $c = a/b$ in lowest terms, then

$$\lim_{L\to\infty} Q_L(a/b) = \frac{(-1)^a}{b^2} \quad (b \text{ odd}), \qquad \lim_{L\to\infty} Q_L(a/b) = 0 \quad (b \text{ even}),$$

and the limit is 0 for irrational c.

Proposition 3.9 (rays through the origin)**.** *Let $p, q \ge 1$ be coprime and n odd. The mean of $s(nu)s(nv)$ along the line $(u, v) = (pw, qw)$, $w \in [0, 1)$, equals $1/(pq)$ if p and q are both odd, and 0 otherwise. The value does not depend on n, hence is also the value for every stack.*

Remark 3.10. The number $1/(pq)$ in Proposition 3.9 is the master integral of Theorem 3.1 evaluated at the coprime pair (p, q): an *uncentred* mean. It is not the correlation of the sheets p and q, which by Theorem 3.4 is exactly 0. The two quantities are computed by one lemma read two ways, and the temptation to call them the same number should be resisted.

Remark 3.11. Theorem 3.8 is a limit, and a finite stack does not sit on it. In the stack of the 28 odd scales up to 55: the $b = 7$ ray sits exactly on its limit $-1/49$ (because $28 = 4 \cdot 7$ samples the residues evenly); the $b = 9$ ray reads $+1/63$ against a limit of $-1/81$, the *wrong sign*; the $b = 5$ ray is $10/7$ of its limit; and the $b = 6$ ray reads $-1/42$ against a limit of 0, though $b = 2$ is exactly 0 at every stack size. Any image of a finite stack is therefore an illustration of these laws, not a measurement of them.

Proposition 3.12 (exact stack variance)**.** *For $S = \{1, 3, \ldots, 2L-1\}$ and the carpet stack $G_L = \frac{1}{L}\sum_{n\in S} \mathrm{C}_n$,*

$$\mathrm{Var}(G_L) = \frac{1}{L^2} \sum_{m,n\in S} \frac{(d^2-1)\big[2(m-1)(n-1) + d^2 - 1\big]}{16 m^2 n^2}, \quad d = \gcd(m, n),$$

an exact rational for every L. Its values are $L \cdot \mathrm{Var} = 0.21014$ at $L = 28$, 0.24516 at $L = 101$ and 0.26400 at $L = 501$.

Conjecture 3.13. $\lim_{L\to\infty} L \cdot \mathrm{Var}(G_L) = 0.272442\ldots$, *and this constant is a combination of two gcd sums rather than a known closed form. Evidence: the exact values* 0.21014, 0.24516 *and* 0.26400 *of Proposition 3.12 climb towards it, and the limit reduces to the two gcd sums*

$$S_2(N) = \sum g^2/(mn), \quad S_4(N) = \sum g^4/(m^2n^2) \quad (m, n \text{ odd}, \le N)$$

as $\lim_{N\to\infty} S_2(N)/(4N) + \lim_{N\to\infty} S_4(N)/(8N)$. *The second of these reduces to a single gcd sum,*

$$\lim_{N\to\infty} \frac{S_4(N)}{N} = \frac{16}{31} \cdot \frac{T}{\zeta(5)} = 0.5536124372\ldots,$$
$$T = \sum_{k,l \ \text{odd}} \frac{1}{k^2 l^2 \max(k,l)} = 1.1122336970\ldots,$$

so the second term contributes $0.06920155\ldots$ *and the first carries the rest. Failure modes: no closed form is known for T, so the constant is* reduced, *not evaluated; the reduction's error term is only measured, at five digits and still drifting; and the finite-L approach is slow enough (*23% *short at* $L = 28$*) that agreement at accessible L is weak evidence for any particular closed form. Nothing in this paper depends on this conjecture.*

3.1. **The spectral measure of the stack.** The correlation laws above are diagonal statistics; this section records the whole spectrum. The basis is the product sine system $\{2\sin(\pi au)\sin(\pi bv)\}_{a,b\ge1}$, orthonormal and complete on the square.

Theorem 3.14 (spectral measure of the stack)**.** *Let G_L be the stack of Definition 2.3 and write $\sigma_k^S(a) = \sum_{n\in S,\ n|a} n^k$ for the divisor sums restricted to the stack's scales. For odd $a, b \ge 1$,*

$$\int_0^1\int_0^1 G_L(u,v)\ \sin(\pi au)\sin(\pi bv)\,du\,dv$$
$$= \frac{1}{\pi^2 ab}\Big[1 - \frac{\sigma_1^S(a)}{L} - \frac{\sigma_1^S(b)}{L} + \frac{\sigma_2^S(\gcd(a,b))}{L}\Big],$$

and the integral vanishes whenever a or b is even. For $\gcd(a,b) \le 2L-1$ the divisor sums are complete: $\sigma_2^S(\gcd(a,b)) = \sigma_2(\gcd(a,b))$, the full sum of squared divisors. In particular the doubly-oscillatory component of the stack carries the coefficient field $\sigma_2(\gcd(a,b))/(ab)$ on odd frequency pairs: a pure point spectrum whose weights are squared-divisor sums of the frequency gcd.

Corollary 3.15 (the variance splits into its spectral blocks)**.** *Write $\widehat{s}_n = s_n - 1/n$ (the centred wave; $\langle s_n\rangle = 1/n$ for odd n by Corollary 3.2) and $\alpha_n = 1 - 1/n$, so that*

$$\mathrm{C}_n = \tfrac14\big[\alpha_n^2 - \alpha_n\widehat{s}_n(u) - \alpha_n\widehat{s}_n(v) + \widehat{s}_n(u)\widehat{s}_n(v)\big]$$

decomposes $G_L - \langle G_L \rangle$ into three mutually orthogonal blocks: two axis blocks and one interior block. Parseval then evaluates the paper's exact variance (Proposition 3.12) blockwise, with $d = \gcd(m,n)$:

$$\mathrm{Var}(G_L) = \underbrace{\frac{2}{16L^2}\sum_{m,n\in S}\frac{(m-1)(n-1)(d^2-1)}{m^2n^2}}_{\textit{axis blocks}} + \underbrace{\frac{1}{16L^2}\sum_{m,n\in S}\frac{(d^2-1)^2}{m^2n^2}}_{\textit{interior block}},$$

which is exactly the two-term split of the carpet covariance law of Theorem 3.3.

Theorem 3.16 (zeta quotients)**.** *Let $\lambda(w) = (1-2^{-w})\,\zeta(w)$, the odd-index zeta. Then, with all sums over odd $a, b \geq 1$ and $g = \gcd(a,b)$:*

$$\sum_{a,b}\frac{\sigma_2(g)}{(ab)^w} = \lambda(w)^2\,\lambda(2w-2) \qquad (\mathrm{Re}\,w > \tfrac{3}{2}),$$

$$\sum_{a,b}\frac{\sigma_2(g)^2}{(ab)^w} = \frac{\lambda(w)^2\,\lambda(2w-2)^2\,\lambda(2w-4)}{\lambda(4w-4)} \qquad (\mathrm{Re}\,w > \tfrac{5}{2}).$$

The second identity is Ramanujan's $\sum\sigma_a(n)\sigma_b(n)n^{-s}$ evaluation at $a = b = 2$, restricted to odd integers, combined with the gcd decoupling of the first. Every polynomial spectral statistic of the stack is such a finite product and quotient of zeta values at integer-shifted arguments.

Corollary 3.17 (weighted stacks render divisor functions)**.** *Weight layer n by n^{-s} instead of $1/L$: $G^{(s)} = \sum_{n\geq 1\ odd} n^{-s}\,\mathrm{C}_n$, convergent pointwise a.e. and in L^2 for $\mathrm{Re}\,s > 1$. Its interior spectral coefficients are σ_{2-s}-divisor sums of the frequency gcd: the interior coefficient at odd (a,b) is $(\pi^2 ab)^{-1}\sum_{n|\gcd(a,b)} n^{2-s}$ (odd divisors), each an entire function of s, and*

$$\sum_{a,b\ odd}\frac{\sigma_{2-s}(\gcd(a,b))}{(ab)^w} = \lambda(w)^2\,\lambda(2w+s-2).$$

A weighted stack is a Dirichlet series rendered as a picture: the weights of its spectrum are the coefficients of $\lambda(2w+s-2)$'s Euler product, frequency pair by frequency pair.

Remark 3.18 (what the spectrum does and does not carry)*.* The stack's spectral measure is classical divisor-sum content and nothing beyond: every statistic in Theorem 3.16 lies in the Estermann–Ramanujan class of zeta quotients, the same class as $\sum\sigma_a(n)\sigma_b(n)n^{-s}$. Their meromorphic continuations behave accordingly – the quotient in the second identity has $\lambda(4w-4)$ in its denominator, so its continuation past the natural half-plane inherits the zeros of $\zeta(4w-4)$, exactly as Ramanujan's identity always does; nothing here sharpens or evades that classical sensitivity, and no new L-function appears. This is the spectral companion of Proposition 3.21: the brightness carries no Möbius information, and the spectrum carries divisor information only.

3.2. **Where the exactness stops.**

Proposition 3.19 (the space rule has no pair term)**.** *Write $\sigma_i = +1$ if index i is even and -1 if it is odd. The plane rule "ink unless both indices are odd" has the harmonic expansion*

$$H(\sigma_1, \sigma_2) = \tfrac{3}{4} + \tfrac{\sigma_1+\sigma_2}{4} - \tfrac{\sigma_1\sigma_2}{4},$$

while the space rule "fill if at most one of three indices is odd" has

$$F(\sigma_1, \sigma_2, \sigma_3) = \tfrac{1}{2} + \tfrac{\sigma_1+\sigma_2+\sigma_3}{4} - \tfrac{\sigma_1\sigma_2\sigma_3}{4},$$

whose three pairwise coefficients are all exactly 0.

The pair term $-\sigma_1\sigma_2/4$ is the entire mechanism of this paper: it is what turns the master integral into a correlation. In space the pair term is absent and a triple product takes its place, so a planar section of the space rule has no reason to inherit any of the exactness – and it does not.

Remark 3.20. On rendered hexagonal sections of the space rule at the 28 scales $1, 3, \ldots, 55$, the 290 coprime pairs drawn from the 27 scales $3, \ldots, 55$ have Pearson correlations with mean -0.037 and range $[-0.205, +0.147]$, with 85 of the 290 exceeding 0.05 in absolute value; the corresponding flat-carpet figures on the same raster are mean -0.0001 and maximum 0.017, against exactly 0 in the continuum. The gcd echo survives the passage to the hexagonal section; the exactness does not. These are measurements on rendered images, not theorems, and they inherit the usual caveats: the extreme values are mask-dependent, and the same 290 pairs read mean -0.009 over the range $[-0.173, +0.160]$ on a coarser common raster with a differently registered hexagon. What is robust across every mask tried is the shape of the failure, not the size of the worst value. They are reported here because a negative result with soft numbers is still worth more than silence, and they are outside the domain that `scripts/verify.py` checks.

Proposition 3.21 (the picture cannot see Möbius)**.** *Let $K(a/b)$ denote the ray strength of Theorem 3.8, so $K(a/b) = (-1)^a/b^2$ for odd b and $K(a/b) = 0$ for even b. There is no function f with $\mu(b) = f\big(K(a/b)\big)$ for all reduced a/b, where μ is the Möbius function.*

The consequence is worth stating in words, because the temptation runs the other way. This stack draws a bright line at every rational with odd denominator and nothing at the others; its brightnesses index denominators and nothing else. Squarefreeness is not visible in it, so no Báez-Duarte-style criterion for the Riemann hypothesis [2] can be extracted from these brightnesses. Any such computation gets its arithmetic from a factorisation performed elsewhere, and the picture contributes nothing to it.

4. Proofs

Proof of Theorem 3.1. The system $\{\sin(\pi cu)\}_{c\geq 1}$ is orthogonal on $(0,1)$ with

$$\int_0^1 \sin(\pi au)\sin(\pi bu)\,du = \tfrac{1}{2}\delta_{ab} \qquad (a, b \geq 1 \text{ integers}),$$

since $2\sin(\pi au)\sin(\pi bu) = \cos(\pi(a-b)u) - \cos(\pi(a+b)u)$, and $\int_0^1 \cos(\pi cu)\,du$ equals $\sin(\pi c)/(\pi c)$, which vanishes for every nonzero integer c; here $a+b \neq 0$ always, while $a-b=0$ contributes 1. It is moreover a complete orthogonal system for $L^2(0,1)$, being the Fourier sine basis.

The square wave has the classical expansion $s(x) = \frac{4}{\pi}\sum_{k \text{ odd}} \frac{\sin(\pi kx)}{k}$, convergent in L^2 of any bounded interval [5]. Substituting $x = mu$ gives, in $L^2(0,1)$,

$$s(mu) = \frac{4}{\pi}\sum_{k \text{ odd}} \frac{\sin(\pi kmu)}{k},$$

which is already an expansion in the above system: the coefficient attached to the basis index c is $4/(\pi k)$ when $c = km$ with k odd, and 0 for every other c (distinct odd k give distinct c). Parseval for two functions in an orthogonal system with $\|\sin(\pi c\cdot)\|^2 = \frac{1}{2}$ therefore gives

$$\int_0^1 s(mu)s(nu)\,du = \frac{1}{2}\sum_{c\geq 1}\Big(\frac{4}{\pi j}\Big)\Big(\frac{4}{\pi k}\Big) = \frac{8}{\pi^2}\sum_{\substack{j,k \text{ odd} \\ jm=kn}} \frac{1}{jk},$$

the inner sum running over the pairs (j,k) of odd positive integers with $jm = kn = c$.

Now solve $jm = kn$. Dividing by g gives $jm' = kn'$ with $\gcd(m', n') = 1$, so $n' \mid j$; writing $j = n't$ forces $k = m't$, and conversely every $t \geq 1$ gives a solution. The constraint that j and k be odd says exactly that $n't$ and $m't$ are odd, i.e. that m', n' and t are all odd. If m' or n' is even there is no admissible pair and the sum is empty, giving 0. Otherwise

$$\frac{8}{\pi^2}\sum_{t \text{ odd}} \frac{1}{m'n't^2} = \frac{8}{\pi^2}\cdot\frac{1}{m'n'}\cdot\frac{\pi^2}{8} = \frac{1}{m'n'} = \frac{g^2}{mn},$$

using $\sum_{t \text{ odd}} t^{-2} = (1-\frac{1}{4})\zeta(2) = \pi^2/8$. □

Proof of Corollary 3.2. If m, n are odd so are m' and n'. For the mean, take $m = 1$: $s(u) = 1$ on $[0,1)$, so $\langle s(nu)\rangle = \int_0^1 s(1\cdot u)s(nu)\,du = 1/n$ for odd n by Theorem 3.1. Then $\langle \chi_n\rangle = \frac{1}{2}(1-\frac{1}{n}) = \frac{n-1}{2n}$. □

Proof of Theorem 3.3. Write $a_m = 1/m$ and $\mu_m = \langle\chi_m\rangle = (1-a_m)/2$, put $A = \int_0^1 s(mu)s(nu)\,du = d^2/(mn)$, and set

$$\nu = \langle\chi_m\chi_n\rangle = \tfrac{1}{4}\langle(1-s(mu))(1-s(nu))\rangle = \tfrac{1}{4}(1-a_m-a_n+A),$$

all by Corollary 3.2 and Theorem 3.1. Note $\nu - \mu_m\mu_n = \frac{1}{4}(A - a_m a_n) = \frac{d^2-1}{4mn}$.

Tree. $\mathrm{Cov}(\mathrm{T}_m, \mathrm{T}_n) = \nu - \mu_m\mu_n$, and $\mathrm{Var}(\mathrm{T}_n) = \mu_n(1-\mu_n) = \frac{n^2-1}{4n^2}$ because χ_n is $\{0,1\}$-valued. Dividing gives the stated r.

Carpet. Since u and v are independent coordinates and $\mathrm{C}_n = \chi_n(u)\chi_n(v)$, we get $\langle \mathrm{C}_m\mathrm{C}_n\rangle = \nu^2$ and $\langle \mathrm{C}_n\rangle = \mu_n^2$, so $\mathrm{Cov} = \nu^2 - \mu_m^2\mu_n^2 = (\nu - \mu_m\mu_n)(\nu + \mu_m\mu_n)$. Now $4\nu = 1 - a_m - a_n + A$ and $4\mu_m\mu_n = 1 - a_m - a_n + a_m a_n$, so

$$\nu + \mu_m\mu_n = \tfrac{1}{4}(2 - 2a_m - 2a_n + A + a_m a_n) = \frac{2(m-1)(n-1) + d^2 - 1}{4mn},$$

using $2 - 2a_m - 2a_n + 2a_m a_n = 2(1 - a_m)(1 - a_n)$ and $A - a_m a_n = (d^2 - 1)/(mn)$. Multiplying by $\nu - \mu_m\mu_n = (d^2 - 1)/(4mn)$ gives the stated covariance. For the variance put $m = n$, $d = n$: $\mathrm{Var}(\mathrm{C}_n) = \mu_n^2(1 - \mu_n^2) = \frac{(n-1)^2(n+1)(3n-1)}{16n^4}$, since $4n^2 - (n-1)^2 = (n+1)(3n-1)$. Dividing the covariance by $\sqrt{\mathrm{Var}(\mathrm{C}_m)\,\mathrm{Var}(\mathrm{C}_n)}$ gives the stated r.

Net. $\mathrm{N}_n = (1 - \chi_n(u))(1 - \chi_n(v))$ has $\langle \mathrm{N}_n\rangle = \bar\mu_n^2$ with $\bar\mu_n = (1 + a_n)/2$, and $\langle \mathrm{N}_m\mathrm{N}_n\rangle = \bar\nu^2$ with $\bar\nu = 1 - \mu_m - \mu_n + \nu = \frac{1}{4}(1 + a_m + a_n + A)$. This is the carpet computation with a_m, a_n replaced by $-a_m, -a_n$ and A unchanged, so the same algebra returns the same expression with $(m-1)(n-1)$ replaced by $(m+1)(n+1)$. For the variance put $m = n$, $d = n$: $\mathrm{Var}(\mathrm{N}_n) = \bar\mu_n^2(1 - \bar\mu_n^2) = \frac{(n+1)^2(n-1)(3n+1)}{16n^4}$, since $4n^2 - (n+1)^2 = (n-1)(3n+1)$. Dividing the covariance by $\sqrt{\mathrm{Var}(\mathrm{N}_m)\,\mathrm{Var}(\mathrm{N}_n)}$ gives the stated r.

Void. $\mathrm{V}_n = \frac{1}{2}(1 + s(nu)s(nv))$ gives $\langle \mathrm{V}_n\rangle = \frac{1}{2}(1 + a_n^2)$ and

$$\langle \mathrm{V}_m\mathrm{V}_n\rangle = \tfrac{1}{4}(1 + a_m^2 + a_n^2 + A^2),$$

because the coordinates separate and $\langle s(mu)s(nu)\rangle \cdot \langle s(mv)s(nv)\rangle = A^2$. Hence $\mathrm{Cov} = \frac{1}{4}(A^2 - a_m^2 a_n^2)$, which is $\frac{d^4-1}{4m^2n^2}$. Since V_n is $\{0,1\}$-valued, $\mathrm{Var}(\mathrm{V}_n) = \langle \mathrm{V}_n\rangle(1 - \langle \mathrm{V}_n\rangle) = \frac{n^4-1}{4n^4}$, and dividing gives the stated r. □

Proof of Theorem 3.4. Each covariance carries the factor $d^2 - 1$ or $d^4 - 1$, which vanishes at $d = 1$; so all four are 0 when $\gcd(m, n) = 1$. If $d > 1$ then $d^2 - 1 > 0$ and $d^4 - 1 > 0$, and the remaining factors are strictly positive for odd $m, n \geq 3$: $4mn > 0$, $4m^2n^2 > 0$, and both brackets $2(m\mp1)(n\mp1) + d^2 - 1$ are sums of positive terms, since $m - 1 \geq 2$ and $n - 1 \geq 2$. Hence all four are strictly positive. □

Proof of Corollary 3.5. If n is prime then $\gcd(m, n) = 1$ for every m with $3 \leq m < n$, so every covariance vanishes by Theorem 3.4. If n is odd and composite, let p be its least prime factor; then p is odd, $p \geq 3$, and $p \leq \sqrt{n} < n$, so $m = p$ is an odd scale in range with $\gcd(p, n) = p > 1$, and the covariance at (p, n) is strictly positive by Theorem 3.4. □

Proof of Theorem 3.8. Fix x and set $\Phi(y) = s(y)s(y + x)$. Since $s(y + 1) = -s(y)$, we have $\Phi(y + 1) = \Phi(y)$: the product has period 1, even though each factor has period 2. Put $T(x) = \int_0^1 \Phi(y)\,dy$. For odd n, substituting $y = nu$ and using periodicity,

$$\int_0^1 s(nu)s(nu + nc)\,du = \frac{1}{n}\int_0^n \Phi(y)\,dy = T(nc).$$

Next, T has an absolutely convergent cosine expansion. Expand $\sin(\pi k(y+x))$ by the addition formula in the series for s, and average over a period: the mean of $\sin(\pi ky)\sin(\pi jy)$ is $\frac{1}{2}\delta_{jk}$, and the mean of $\sin(\pi ky)\cos(\pi jy)$ is 0 for j and k both odd, since $2\sin(\pi ky)\cos(\pi jy) = \sin(\pi(k+j)y)+\sin(\pi(k-j)y)$ with $k \pm j$ even and $\int_0^1 \sin(\pi cy)\,dy = (1-\cos\pi c)/(\pi c) = 0$ for every even $c \neq 0$, while $c = 0$ gives the zero function. So Parseval gives

$$T(x) = \frac{8}{\pi^2}\sum_{k \text{ odd}} \frac{\cos(\pi kx)}{k^2},$$

absolutely convergent and uniform in x.

Now average over odd n. For a fixed integer $k \geq 1$ and real $\theta = kc$,

$$\frac{1}{L}\sum_{n\in S}\cos(\pi n\theta) \longrightarrow \begin{cases} (-1)^\theta, & \theta \in \mathbb{Z}, \\ 0, & \theta \notin \mathbb{Z}, \end{cases}$$

because for integer θ and odd n we have $\cos(\pi n\theta) = (-1)^{n\theta} = (-1)^\theta$, while for $\theta \notin \mathbb{Z}$ the sum $\sum_{n\in S} e^{i\pi n\theta}$ is geometric with ratio $e^{2i\pi\theta} \neq 1$, hence bounded by $2/|1-e^{2i\pi\theta}|$ uniformly in L, so its average tends to 0. Since $\sum_k k^{-2} < \infty$ and each term is bounded by 1, truncating the sum at $k \leq K$ with an error at most $\frac{8}{\pi^2}\sum_{k>K} k^{-2}$ uniformly in L justifies exchanging the limit with the sum. Therefore

$$\lim_{L\to\infty} Q_L(c) = \frac{8}{\pi^2}\sum_{\substack{k \text{ odd} \\ kc\in\mathbb{Z}}} \frac{(-1)^{kc}}{k^2}.$$

For irrational c the index set is empty and the limit is 0. For $c = a/b$ reduced, $kc \in \mathbb{Z}$ forces $b \mid k$; if b is even this is incompatible with k odd and the limit is 0. If b is odd, write $k = bk'$ with k' odd; then $kc = k'a$ and $(-1)^{k'a} = (-1)^a$, so the sum is $\frac{8}{\pi^2}\frac{(-1)^a}{b^2}\sum_{k' \text{ odd}} k'^{-2} = \frac{(-1)^a}{b^2}$. ☐

Proof of Proposition 3.9. Along the line, $u = pw$ and $v = qw$, so the mean in question is $\int_0^1 s(npw)s(nqw)\,dw$. Apply Theorem 3.1 to the pair (np, nq): its gcd is $n\gcd(p,q) = n$, and the two quotients are p and q. Hence the integral is $n^2/(np\cdot nq) = 1/(pq)$ when p and q are both odd, and 0 otherwise. Neither answer involves n, so averaging over any set of odd scales returns the same value. ☐

Proof of Proposition 3.12. $\mathrm{Var}(G_L) = \frac{1}{L^2}\sum_{m,n\in S}\mathrm{Cov}(\mathrm{C}_m, \mathrm{C}_n)$ by bilinearity, and each summand is given by Theorem 3.3; the diagonal terms $m = n$ are covered by the same formula, which at $d = m = n$ reduces to $\mathrm{Var}(\mathrm{C}_n)$, and the terms with $n = 1$ vanish since $\mathrm{C}_1 \equiv 0$. Every summand is rational, so the total is. The three quoted values are computed exactly in `scripts/verify.py`. ☐

Proof of Theorem 3.14. Two one-dimensional integrals do all the work:

$$\int_0^1 \sin(\pi a u)\,du = \frac{1-\cos\pi a}{\pi a},$$
$$\int_0^1 \sin(\pi j n u)\sin(\pi a u)\,du = \tfrac{1}{2}\mathbf{1}[jn = a].$$

The first is $2/(\pi a)$ for odd a and 0 for even a; the second holds for positive integers jn and a. Expanding the square wave in its odd-harmonic series $s(x) = \frac{4}{\pi}\sum_{j\text{ odd}} \sin(\pi j x)/j$, valid in L^2, and applying the second integral to each term,

$$\int_0^1 s(nu)\,\sin(\pi a u)\,du = \frac{2}{\pi}\,\frac{n}{a}\,\mathbf{1}\big[\,n \mid a \text{ and } a/n \text{ odd}\,\big].$$

For odd a and odd n the cofactor condition is automatic. Now $\mathrm{C}_n = \frac{1}{4}\big(1 - s_n(u)\big)\big(1 - s_n(v)\big)$ splits the double integral into four products of the one-dimensional integrals just computed: for odd a, b,

$$\int_0^1\!\!\int_0^1 \mathrm{C}_n \sin(\pi a u)\sin(\pi b v)\,du\,dv$$
$$= \frac{1}{\pi^2 ab}\Big[1 - n\,\mathbf{1}[n \mid a] - n\,\mathbf{1}[n \mid b] + n^2\,\mathbf{1}[n \mid \gcd(a,b)]\Big].$$

Averaging over $n \in S$ assembles the bracket of the theorem with σ_1^S and σ_2^S. For even a every term vanishes: the constant and the pure-v terms because $\int_0^1 \sin(\pi a u)\,du = 0$, the others because $jn = a$ is impossible with j, n odd. When $\gcd(a,b) \le 2L-1$ every divisor of the odd number $\gcd(a,b)$ is odd and at most $2L-1$, hence lies in S. □

Proof of Corollary 3.15. $\langle s_n\rangle = 1/n$ for odd n (Corollary 3.2), so $\widehat{s}_n = s_n - 1/n$ has mean zero, and $1 - s_n = \alpha_n - \widehat{s}_n$ gives the stated four-term form of C_n. The three families – constants, mean-zero functions of one variable, and products of two mean-zero one-variable functions – are mutually orthogonal in L^2 of the square, because every cross-pairing contains a factor $\langle\widehat{s}_n\rangle = 0$. Within the axis blocks, $\langle\widehat{s}_m\widehat{s}_n\rangle = \langle s_m s_n\rangle - 1/(mn) = (d^2-1)/(mn)$ by Theorem 3.1; within the interior block, $\langle\widehat{s}_m(u)\widehat{s}_m(v)\,\widehat{s}_n(u)\widehat{s}_n(v)\rangle = \langle\widehat{s}_m\widehat{s}_n\rangle^2 = (d^2-1)^2/(mn)^2$. Hence

$$\mathrm{Var}(G_L) = \frac{2}{16L^2}\sum_{m,n\in S}\alpha_m\alpha_n\frac{d^2-1}{mn} + \frac{1}{16L^2}\sum_{m,n\in S}\frac{(d^2-1)^2}{m^2n^2},$$

and $\alpha_m\alpha_n/(mn) = (m-1)(n-1)/(m^2n^2)$ gives the display. Adding the two summands recovers $(d^2-1)[2(m-1)(n-1)+d^2-1]/(16m^2n^2)$, the carpet law of Theorem 3.3, so this is a second, spectral derivation of Proposition 3.12. □

Proof of Theorem 3.16. For the first identity, write $\sigma_2(g) = \sum_{d|g} d^2$ and exchange sums: with all variables odd,

$$\sum_{a,b} \frac{\sigma_2(\gcd(a,b))}{(ab)^w} = \sum_d d^2 \sum_{d|a,\ d|b} (ab)^{-w}$$
$$= \sum_d d^{2-2w} \Big(\sum_{a'} a'^{-w}\Big)^2 = \lambda(2w-2)\,\lambda(w)^2,$$

using that $a = da'$ ranges over odd multiples exactly when d and a' are odd. Convergence needs $\mathrm{Re}\, w > 1$ and $\mathrm{Re}(2w-2) > 1$, i.e. $\mathrm{Re}\, w > 3/2$. For the second, two classical steps. Step 1: $E(s) := \sum_{n \text{ odd}} \sigma_2(n)^2 n^{-s} = \lambda(s)\lambda(s-2)^2\lambda(s-4)/\lambda(2s-4)$. This is Ramanujan's identity $\sum_n \sigma_a(n)\sigma_b(n)n^{-s} = \zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)/\zeta(2s-a-b)$ at $a = b = 2$ [4, 6], restricted to odd n: both sides are Euler products over primes, the local factor at p being $(1-p^{4-2s})\big[(1-p^{-s})(1-p^{2-s})^2(1-p^{4-s})\big]^{-1}$, so deleting the $p = 2$ factor replaces each ζ by λ. Step 2: for any multiplicative f on the odd integers, $\sum_{a,b \text{ odd}} f(\gcd(a,b))(ab)^{-w} = \lambda(w)^2 \sum_{g \text{ odd}} (f * \mu)(g)\, g^{-2w} = \lambda(w)^2\, D_f(2w)/\lambda(2w)$, where D_f is the Dirichlet series of f over odd integers, by Möbius inversion of the divisor-sum exchange in the first display. Taking $f = \sigma_2^2$, $D_f = E$, gives $\lambda(w)^2 E(2w)/\lambda(2w) = \lambda(w)^2\lambda(2w-2)^2\lambda(2w-4)/\lambda(4w-4)$. Convergence needs $\mathrm{Re}(2w-4) > 1$, i.e. $\mathrm{Re}\, w > 5/2$. □

Proof of Corollary 3.17. For $\mathrm{Re}\, s > 1$ the layer sum converges absolutely, since $0 \le \mathrm{C}_n \le 1$. Exactly as in the proof of Theorem 3.14, the interior coefficient of layer n is $(\pi^2 ab)^{-1} n^2 \mathbf{1}[n \mid \gcd(a,b)]$, so weighting by n^{-s} and summing over odd n gives $(\pi^2 ab)^{-1} \sum_{n|\gcd(a,b)} n^{2-s}$; for odd a and b every divisor qualifies, and the sum is finite, entire in s. The Dirichlet identity is the first display of the proof of Theorem 3.16 with d^2 replaced by d^{2-s}. □

Proof of Proposition 3.19. A function on $\{\pm 1\}^k$ has a unique multilinear expansion whose coefficients are the correlations with the corresponding monomials. For H: the value is 0 only at $(\sigma_1, \sigma_2) = (-1,-1)$ and 1 elsewhere, so the constant term is 3/4, each single coefficient is $\frac{1}{4}(1+1-1+0) = \frac{1}{4}$, and the pair coefficient is $\frac{1}{4}(1-1-1+0) = -\frac{1}{4}$. For F: the value is 1 at the four patterns with at most one -1 and 0 at the other four, so the constant term is 1/2; each single coefficient is $\frac{1}{8}(1-1+1+1) = \frac{1}{4}$; each pair coefficient is $\frac{1}{8}(1-1-1+1) = 0$; the triple coefficient is $\frac{1}{8}(1-1-1-1) = -\frac{1}{4}$. Both expansions are verified by exhaustive evaluation in `scripts/verify.py`. □

Proof of Proposition 3.21. Take b even. Then $K(a/b) = 0$ for every reduced a/b, while μ takes all three values $-1, 0, +1$ on even b: $\mu(2) = -1$, $\mu(4) = 0$, $\mu(6) = +1$. A function of $K(a/b)$ is constant on that set and μ is not, so no such f exists. □

5. Reproducibility

Two scripts, both plain `python3` with only the standard library, both run from the paper's own directory, no arguments and no network. Together they take under two seconds on a laptop.

`python3 scripts/verify.py` runs in about 1.2 seconds and prints one line per block, ending in `all green`; every check is an assertion naming the value it got and the value it wanted, and exit code 0 is the only pass signal. It checks, in order: (i) Theorem 3.1 against an exact rational integration on the lcm grid for *all* pairs $1 \le m \le n \le 40$, zero mismatches, and confirms that the unqualified law $g^2/(mn)$ fails on exactly 532 of those pairs, with $(1,2) \mapsto 0$ and $(2,6) \mapsto 1/3$; (ii) the same integration for all odd pairs $1 \le m \le n \le 99$, zero mismatches, and $\langle s(mu) \rangle = 1/m$ for every odd $m \le 99$; (iii) the carpet, net and void covariances by genuine two-dimensional cell counting on the lcm $\times$ lcm grid at $(3,9), (5,15), (9,15), (7,21), (3,5), (5,7)$, matching Theorem 3.3 exactly, with $\mathrm{Cov}(\mathrm{C}_3, \mathrm{C}_9) = 20/729$ and $\mathrm{Cov}(\mathrm{N}_3, \mathrm{N}_9) = 44/729$, and the net's Pearson formula against the ratio built from those same counts; (iv) all four closed forms of Theorem 3.3 against covariances assembled from the brute-forced integrals, for every odd pair $3 \le m \le n \le 39$, together with Theorem 3.4 on that domain, where exactly 139 pairs are coprime, and the four correlations quoted at $(3,9)$ in their stated order; (v) Fact 3.6 in full over odd $3 \le n \le 199$, each of the three quoted scores at $n = 169$ recomputed as a true maximum over all odd $m < 169$; (vi) Proposition 3.9 for every odd $n \le 9$ and all coprime $p, q \le 8$; (vii) the finite-stack ray values of Remark 3.11 at $L = 28$ in exact rationals; (viii) Proposition 3.12 at $L = 28, 101, 501$, the whole double sum accumulated in exact rational arithmetic and all three values asserted; (ix) the classical sawtooth identity of Section 1.1 for all pairs $1 \le m \le n \le 16$; (x) the truncation of T at cutoff 1001, giving 1.112233083 against the quoted 1.1122336970; (xi) Proposition 3.21 over all 4081 reduced a/b with even $b \le 200$, the vanishing ray strength of each recomputed independently of the formula, as the exact triangle-wave stack average $\frac{1}{b} \sum_n T(na/b)$ over the b odd scales $n < 2b$, which is 0 because $n \mapsto n + b$ pairs the terms with opposite signs; and (xii) Proposition 3.19 by exhaustive evaluation of both rules. Four blocks cover the spectral section: (xiii) Theorem 3.14 at $L = 5$ by cell-exact evaluation of the layer integrals $\int_0^1 \chi_n \sin(\pi a u)\, du$ against the divisor-sum formula for all pairs $a, b \le 15$ and at $(45, 45)$, with every even-indexed coefficient vanishing to 10^{-12}; (xiv) the blockwise variance split of Corollary 3.15 as an exact rational identity for $L = 2$ through 8; (xv) both zeta-quotient identities of Theorem 3.16, the truncated double gcd-sums at $w = 3$ and $w = 3.5$ against the λ-products with ζ evaluated by Euler–Maclaurin; and (xvi) the weighted-stack identity of Corollary 3.17 at $s = 3$, $w = 2$.

`python3 scripts/figure.py` runs in well under a second and writes `figures/carpet.svg`, the correlation matrix $r(\mathrm{C}_m, \mathrm{C}_n)$ over odd $3 \le m, n \le 63$ shaded from white at 0 to black at 1, with the prime scales labelled; the white rows and columns are the primes. It is the only figure the repository's `README.md` embeds; Fig. 1 is drawn inline in the manuscript.

Two claims in this paper are deliberately outside the scripts' reach and are labelled as such where they appear. Remark 3.20 reports correlations measured on rendered hexagonal sections, which needs a rasteriser and is a measurement, not a theorem. Conjecture 3.13 is a conjecture about a limit; only its exact finite-L values are checked. Everything else in the paper is either proved in Section 4 or checked by `scripts/verify.py` on the exact domain stated in its own statement. The manuscript builds with `tectonic paper.tex`.

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