MENGER DIAGONAL SLICES: A RECURRENCE OF ORDER $\lceil D/2 \rceil$

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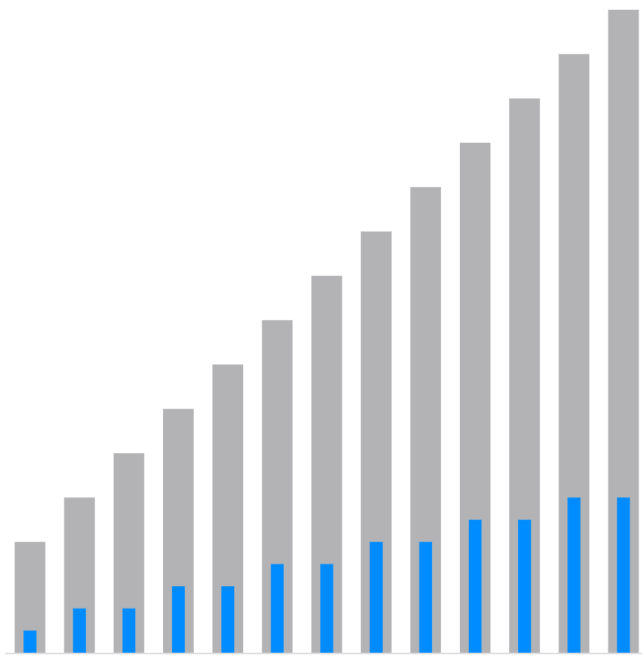

Abstract. Slice a Menger sponge along its main diagonal and count the cubes the cut meets; then do the same in every dimension. The count obeys a linear recurrence with constant coefficients of order at most $\lceil D/2 \rceil$ in dimension D, a fourfold sharpening of the order $2D+1$ that the standard carry automaton gives for free; exactness of that order is verified for $2 \le D \le 24$. A discrete Fourier extraction then turns a relaxed census into an exact product of cosines, and the formula decides half of the sign law this cut obeys: at every odd dimension D with $D \not\equiv 1 \pmod 3$, the counting exponent sits strictly above the solid's dimension minus one, proved, with the parity carried by the single factor $(-1)^{D-1}(D-1)$ that every term of the formula ends in. Two exact identities frame what remains: the dominant root is pinned to within $2(D-1)/3$ of $f_D/3$ unconditionally, so the slice exponent converges to the generic Marstrand–Mattila value in every case, and the whole sign law is equivalent to the parity-free statement that the Perron carry vector puts more than a third of its mass on carries divisible by three. At $D = 3$ the machine outputs $\left(\begin{smallmatrix} 6 & 6 \\ 1 & 3 \end{smallmatrix}\right)$, reproducing the published hexagon-triangle substitution spectrum and the integers $1, 6, 42, 306, 2250, 16578, 122202$ by a route that never mentions a hexagon.

1. Introduction

Take a cube. Cut it into 27 equal subcubes and throw away the seven that touch the middle of a face or the middle of the cube; twenty survive. Repeat inside each survivor forever, and the limit is the Menger sponge. Now slice the sponge with the plane through its centre perpendicular to a main diagonal. The picture that falls out is famous: a hexagon full of six-pointed stars, first rendered by Pérez-Duarte [12], popularised by Hart [5] and by Cook's slicing code [3], and given its substitution rule and its dimension 1.8184 by Abel [1].

This paper asks the same question one dimension at a time. In dimension D the sponge becomes the set of cells with at most one coordinate digit equal to the middle digit 1, and the diagonal cut becomes the hyperplane $\sum_i x_i = D(3^L - 1)/2$ through the centre of the level-L grid. Let $a_D(L)$ count the cells the hyperplane meets. The question is how much memory this sequence has: how many previous terms determine the next one.

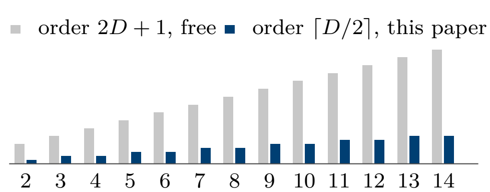

Figure 1. How much memory the slice census needs, by dimension D.

The answer is Theorem 3.1: at most $\lceil D/2 \rceil$ terms, in every dimension at once. Figure 1 is that half of the contribution in one picture; the other half is Theorem 3.18, a proved sign law at odd dimension. Three moves prove the order. First, the digit polynomial factors,

$$P_D(t) = (1+t^2)^{D-1}(1+Dt+t^2),$$

which is a two-line count of how a kept digit vector can sum to s. Second, the carry map $c \mapsto (c+D-s)/3$ contracts: its fixed point is $D/2$, integrality is a wall, and every carry reachable from zero obeys $|c| \le \lfloor (D-1)/2 \rfloor$. Third, the design is invariant under $v \mapsto (2,\dots,2) - v$, so $P_D[s] = P_D[2D-s]$, the carry reflection $c \mapsto -c$ commutes with the transfer matrix, and the orbit of the start state lives inside its $+1$ eigenspace, whose dimension is $\lfloor (D-1)/2 \rfloor + 1 = \lceil D/2 \rceil$.

What is not new. That some constant-coefficient recurrence exists is free and standard. A finite carry state space makes $a_D(L)$ a 3-regular sequence in the sense of Allouche and Shallit [2], and the untrimmed automaton on states $-D,\dots,D$ already hands over order $2D+1$ with no argument at all. A paper that led with “these slices obey a linear recurrence” would be selling a textbook fact. The object is not new either: higher-dimensional Menger continua are classical, and Hocking [6] cuts exactly the $D = 4$ diagonal for its geometry and its pictures. The $D = 3$ integers are catalogued as OEIS A299916 [11], whose Menger reading is a user comment counting the star-shaped holes rather than the tiles. The carry-automaton frame is classical too: with unrestricted digits it is the carries Markov chain of Holte’s “amazing matrix” [7], whose spectrum is known exactly; the design’s joint digit restriction and pinned output digit take the matrix here outside that theory, and the precise relationship is spelled out in [4].

What is new. The exact factorisation of P_D, the sharp order $\lceil D/2 \rceil$ proved for every $D \ge 2$, the exactness of that order verified through $D = 24$, the closed form of the reduced matrix, and every census at $D \ge 4$. Roughly a factor of four, uniformly in the dimension. Then, from the same factorisation read on the unit circle: an exact trigonometric product formula for a relaxed census (Theorem 3.16), and with it the odd half of the sign law – $\rho_D > f_D/3$ for every odd $D \not\equiv 1 \pmod 3$, proved (Theorem 3.18), a statement that had resisted spectral routes because the relevant gap closes like $4/D$ (Conjecture 3.25).

The cold anchor. At $D = 3$ the construction, which never mentions a hexagon or a triangle, returns the 2×2 integer matrix $\left(\begin{smallmatrix} 6 & 6 \\ 1 & 3 \end{smallmatrix}\right)$, characteristic polynomial $\lambda^2 - 9\lambda + 12$, dominant root $(9+\sqrt{33})/2$ and census $1, 6, 42, 306, 2250, 16578, 122202$. Abel’s tile substitution $\left(\begin{smallmatrix} 6 & 1 \\ 6 & 3 \end{smallmatrix}\right)$ [1] has the same trace 9 and determinant 12, A299916 has signature $(9, -12)$ and the same integers, and $\log_3((9+\sqrt{33})/2) = 1.818410$ is Abel’s published dimension. Two different routes to one spectrum is the reason to believe the machine in the dimensions where nobody has looked.

2. Definitions

Definition 2.1. The *D-dimensional Menger design* is

$$F_D = \{v \in \{0,1,2\}^D : \#\{i : v_i = 1\} \le 1\},$$

the digit vectors with at most one middle coordinate. Its *fill* is $|F_D|$.

Definition 2.2. For $L \ge 0$ write each $x_i \in \{0,\dots,3^L-1\}$ in base 3 as $x_i = \sum_{j=0}^{L-1} v_i^{(j)} 3^j$. The *level-$L$ Menger analog* is

$$S_L^{(D)} = \{x : v^{(j)} \in F_D \text{ for every } j\},$$

and the *central diagonal census* is

$$a_D(L) = \#\Big\{x \in S_L^{(D)} : \textstyle\sum_i x_i = D(3^L-1)/2\Big\}, \qquad a_D(0) = 1.$$

The hyperplane is the one through the centre of the grid: the coordinate sum of the centre cell of $\{0,\dots,3^L-1\}^D$ is $D(3^L-1)/2$. At $D=3$ this is the diagonal cut of the classical sponge.

Definition 2.3. The *digit polynomial* is

$$P_D(t) = \sum_{v \in F_D} t^{v_1+\cdots+v_D} = \sum_{s=0}^{2D} B_D(s)\, t^s.$$

Definition 2.4. The *transfer matrix* $M^{(D)}$ acts on carry states $c \in \mathbb{Z}$ by

$$M^{(D)}_{c'c} = B_D(c + D - 3c'),$$

which is nonzero only when $0 \le c + D - 3c' \le 2D$. The *core* is $C_D = \{c \in \mathbb{Z} : |c| \le r_D\}$ with $r_D = \lfloor (D-1)/2 \rfloor$, and $M^{(D)}_{\text{even}}$ is the matrix of $M^{(D)}$ restricted to the core and expressed in the reflection-symmetric basis $v_0 = e_0$, $v_k = e_k + e_{-k}$ for $1 \le k \le r_D$. Its size is $n_D = r_D + 1 = \lceil D/2 \rceil$, and χ_D denotes its characteristic polynomial.

3. Results

Theorem 3.1. *For every $D \ge 2$ the sequence $a_D(L)$ satisfies the linear recurrence with constant integer coefficients given by χ_D, the characteristic polynomial of $M^{(D)}_{\text{even}}$. In particular a_D satisfies a constant-coefficient linear recurrence of order at most $\lceil D/2 \rceil$.*

Lemma 3.2. *For every $D \ge 1$, $P_D(t) = (1+t^2)^{D-1}(1 + Dt + t^2)$, the fill is $|F_D| = 2^{D-1}(D+2)$, and*

$$B_D(2k) = \binom{D}{k}, \qquad B_D(2k+1) = D\binom{D-1}{k},$$

with the convention that a binomial coefficient with an out-of-range lower index is zero.

Lemma 3.3. *Every carry state reachable from $c=0$ lies in the core C_D, and the core is forward invariant: if $|c|\le r_D$ and $M^{(D)}_{c'c}\neq 0$ then $|c'|\le r_D$.*

Lemma 3.4. *$P_D[s]=P_D[2D-s]$ for all s, the reflection $R: e_c\mapsto e_{-c}$ satisfies $RM^{(D)}=M^{(D)}R$, and $Re_0=e_0$. Consequently the orbit $\{(M^{(D)})^L e_0 : L\ge 0\}$ lies in the $+1$ eigenspace of R inside the core, a space of dimension $\lceil D/2\rceil$.*

Lemma 3.5. *Let $\omega=e^{2\pi i/3}$. Then $P_D(\omega)=(-1)^{D-1}\omega^D(D-1)$, so $|P_D(\omega)|=D-1$ for every $D\ge 1$.*

Lemma 3.6. *Write $\varepsilon_D=(D-1)(-1)^{D-1}/3$ and $f_D=2^{D-1}(D+2)$. For $j\in\{0,1,2\}$,*

$$\sum_{s\equiv j \pmod 3} B_D(s)=\begin{cases} f_D/3+2\varepsilon_D, & j\equiv D \pmod 3,\\ f_D/3-\varepsilon_D, & \textit{otherwise.}\end{cases}$$

Proposition 3.7. *Fix an integer offset m with $|m|\le r_D$ and let $a_D(L;m)$ count the cells of $S_L^{(D)}$ on the parallel hyperplane $\sum_i x_i = D(3^L-1)/2+m$. Then $a_D(\cdot\,;m)$ satisfies the same recurrence as a_D, the one given by χ_D, and $a_D(L;m)=a_D(L;-m)$ for every L.*

Remark 3.8. The hypothesis $|m|\le r_D$ is sharp at even D. At $D=4$ the offset $m=2=D/2$ sits exactly on the fixed point of the carry map, never enters the core, and the residual of χ_4 applied to $a_4(\cdot\,;2)$ is the constant 128 rather than zero, for every window checked by `scripts/verify.py`.

Corollary 3.9. $\limsup_L a_D(L)^{1/L}$ *is the modulus of a root of χ_D, hence an algebraic number of degree at most $\lceil D/2\rceil$. Whenever the counting exponent* $\dim_{\text{slice}}(D)=\lim_L \log_3 a_D(L)/L$ *exists, it is* $\log_3$ *of that algebraic number.*

Fact 3.10. For every D with $2\le D\le 24$ the Hankel determinant $H_D=\det\bigl[a_D(i+j+1)\bigr]_{0\le i,j<n_D}$, computed in exact rational arithmetic, is nonzero; hence the minimal order of a_D is exactly $\lceil D/2\rceil$ for those D. The first values are $H_2=2$, $H_3=72$, $H_4=-6336$, $H_5=-1029600000$, $H_6=-62272025640000$, and H_{24} is a nonzero integer of 899 digits. Checked by `scripts/verify.py`.

Proposition 3.11. *For every $D\ge 2$ the set of carry states reachable from $c=0$ is exactly the core C_D, of size $D-1$ at even D and D at odd D. Moreover M_C is irreducible and aperiodic: every state reaches* 0 *in one step,* 0 *loops to itself, and* 0 *reaches every state.*

Fact 3.12. For every D with $2\le D\le 24$,

$$\operatorname{tr} M^{(D)}_{\text{even}}=\begin{cases} 3\cdot 2^{D-2}-1, & D \text{ even},\\ 3D\cdot 2^{D-3}, & D \text{ odd}.\end{cases}$$

The even values are $2, 11, 47, 191, 767, 3071, 12287, \ldots$ and the odd values are $9, 60, 336, 1728, 8448, 39936, \ldots$. Checked by `scripts/verify.py`.

Fact 3.13. At $D=3$, $M^{(3)}_{\text{even}} = \left(\begin{smallmatrix}6&6\\1&3\end{smallmatrix}\right)$ with $\chi_3(\lambda)=\lambda^2-9\lambda+12$, dominant root $(9+\sqrt{33})/2$ and census $1,6,42,306,2250,16578,122202$, matching OEIS A299916 term for term and its signature $(9,-12)$, and $\log_3\bigl((9+\sqrt{33})/2\bigr) = 1.818410$ to six places. At $D=2$ the census is 2^L, of order 1. At $D=4$, $\chi_4(\lambda) = \lambda^2 - 11\lambda - 66$ with dominant root $(11+\sqrt{385})/2$ and census $1,6,132,1848,29040,441408,6772128$. At $D=5$ and $D=6$ the censuses begin $1,30,1000,35700,1321600$ and $1,20,4030,242300,24642700$. Checked by `scripts/verify.py`, which also confirms that three independent generators, brute-force digit enumeration, the substitution product $\prod_{j<L} P_D(t^{3^j})$, and the untrimmed carry automaton on states $-D,\dots,D$, agree term for term for $2\le D\le 7$ and $0\le L\le 4$.

Definition 3.14. The *diagonal sheaf census* is

$$b_D(L) = \#\Bigl\{x\in S_L^{(D)} : \textstyle\sum_i x_i \equiv D(3^L-1)/2 \pmod{3^L}\Bigr\}, \qquad b_D(0)=1:$$

the total census of the central hyperplane together with its translates by multiples of 3^L.

Lemma 3.15. *For every real ψ,*

$$P_D(e^{i\psi}) = e^{iD\psi}\,\Phi_D(\psi), \qquad \Phi_D(\psi) = (2\cos\psi)^{D-1}\,(D+2\cos\psi),$$

and Φ_D is real-valued. For odd D, $\Phi_D(\psi)\ge 0$ for every ψ; for even $D>2$, $\Phi_D(\psi)<0$ exactly when $\cos\psi<0$.

Theorem 3.16. *For every $D\ge 2$ and $L\ge 0$,*

$$b_D(L) = \sum_{c\in C_D} (M_C^L)_{c0} = \frac{1}{3^L}\sum_{m=0}^{3^L-1}\prod_{j=0}^{L-1}\Phi_D\Bigl(\frac{2\pi m 3^j}{3^L}\Bigr).$$

In particular the right-hand side, a sum of 3^L products of cosines, is a positive integer.

Corollary 3.17. *For $k\ge 1$ let $U_k=\{1\le u<3^k : 3\nmid u\}$ and*

$$W_k = \sum_{u\in U_k}\prod_{j=0}^{k-1}\Phi_D\Bigl(\frac{2\pi u 3^j}{3^k}\Bigr).$$

Then $3^L b_D(L) = \sum_{k=0}^{L} f_D^{L-k}\,W_k$ with $W_0=1$, equivalently

$$3\,b_D(k) - f_D\,b_D(k-1) = 3^{1-k}\,W_k \qquad (k\ge 1).$$

Moreover the innermost factor of every product in W_k is $\Phi_D(2\pi u/3)$ with $3\nmid u$, and

$$\Phi_D(2\pi/3) = \Phi_D(4\pi/3) = (-1)^{D-1}(D-1),$$

so $W_k = (-1)^{D-1}(D-1)\,V_k$ with $V_k = \sum_{u\in U_k}\prod_{j=0}^{k-2}\Phi_D(2\pi u3^j/3^k)$.

Theorem 3.18. *Let ρ_D be the spectral radius of $M^{(D)}_{\text{even}}$; it is a root of χ_D, and it is the dominant one. For every odd $D \geq 3$,*

$$\rho_D \;\geq\; \frac{f_D}{3},$$

and if moreover $D \not\equiv 1 \pmod 3$, the inequality is strict:

$$\rho_D \;>\; \frac{f_D}{3} \qquad (D \textit{ odd},\; D \equiv 0, 2 \pmod 3).$$

This is the odd half of Conjecture 3.22 for two of the three residue classes.

Fact 3.19. For the remaining residue class $D \equiv 1 \pmod 3$, the integer $\det(f_D I - 3M^{(D)}_{\text{even}})$ is nonzero for every $D \leq 80$, checked in exact integer arithmetic by `scripts/verify.py`; with Theorem 3.18 this gives $\rho_D > f_D/3$ for every odd $D \leq 80$ with no exception. Two structural notes, both checked on the same sweep: for $D \equiv 1 \pmod 3$ the 3-adic valuation of the determinant is at least n_D, so $\chi_D(f_D/3)$ is a nonzero *integer* there; and its valuation is otherwise erratic $(2, 7, 6, 10, 17, 11, 11, 34, 16, 17, 33, 20, 20$ over $D \equiv 1 \pmod 3$, $4 \leq D \leq 40$, both parities), so no uniform 3-adic argument is in sight for this class.

Theorem 3.20 (the pinning)**.** *For every $D \geq 2$,*

$$\left|\rho_D - \frac{f_D}{3}\right| \;\leq\; \frac{2(D-1)}{3},$$

refined by parity to $f_D/3 - 2(D-1)/3 \leq \rho_D \leq f_D/3 + (D-1)/3$ at even D and $f_D/3 - (D-1)/3 \leq \rho_D \leq f_D/3 + 2(D-1)/3$ at odd D. In particular $\rho_D = (f_D/3)(1 + O(D^2 2^{-D}))$, so

$$\log_3 \rho_D - (\log_3 f_D - 1) \longrightarrow 0 :$$

whatever the sign law does, the counting exponent of the recurrence converges to the generic Marstrand–Mattila value $\log_3 f_D - 1$, and the central diagonal plane becomes asymptotically generic. Only the side *of the approach remains in question, and only at even D.*

Proposition 3.21 (the mass identity)**.** *Let v be the Perron vector of M_C, unique up to scale by Proposition 3.11, and let $p_D = \sum_{c\in C_D,\, 3|c} v_c \,/\, \sum_{c\in C_D} v_c$ be its mass on carries divisible by 3. Then, exactly,*

$$3\rho_D \;=\; f_D + (-1)^{D-1}(D-1)\,(3p_D - 1),$$

$$\textit{i.e.} \qquad \operatorname{sgn}\Big(\rho_D - \frac{f_D}{3}\Big) = (-1)^{D+1} \operatorname{sgn}\Big(p_D - \frac{1}{3}\Big).$$

Consequently the full sign law, both parities at once, is equivalent to the parity-free statement that $p_D > \frac{1}{3}$ for every $D \geq 2$: the Perron carry vector always puts more than a third of its mass on the carries divisible by three.

Conjecture 3.22. *For every even $D \geq 2$, and for every odd $D \equiv 1 \pmod 3$ beyond the verified range,*

$$\operatorname{sgn}(\rho_D - f_D/3) = (-1)^{D+1}, \qquad f_D = 2^{D-1}(D+2).$$

The comparison value $\log_3 f_D - 1$ *is the generic one: for a set of Hausdorff dimension* s *in* $\mathbb{R}^D$*, almost every hyperplane meets it in dimension at most* $s-1$*, with equality on a positive-measure set of translates once* $s > 1$ [9, 10]*, and* $\log_3 f_D$ *is the dimension of the solid. The central diagonal hyperplane is one specific, highly symmetric slice, to which no such theorem applies; together with Theorem 3.18 the conjecture says its counting exponent misses the generic value in every dimension, alternately above and below. Evidence: the sign alternates without exception for* $2 \le D \le 20$ *computed from the roots of* χ_D*, and the exact rational sign of* $\chi_D(f_D/3)$ *alternates as* $(-1)^D$ *for* $2 \le D \le 30$*, both in* `scripts/verify.py`*. The sharpest reduced target on the even side is a two-step census contraction: for even* D*, exact computation finds* $9\, b_D(2j+2) \le f_D^2\, b_D(2j)$ *and* $3\, b_D(2j+1) < f_D\, b_D(2j)$ *with no violation through index* 400 *for every even* $D \le 50$*, with the ratio* $-W_{k+1}/(f_D W_k)$ *at odd* k *peaking near* $(D-2)/(D+2)$*, a uniform margin. By induction these two inequalities give* $b_D(L) \le (f_D/3)^L$*, which with Proposition 3.11 (irreducibility) and the nonvanishing of Fact 3.19 would give the even half; neither inequality is proved. Equivalently, by Proposition 3.21, everything left of the law – the even half and the residue class* $D \equiv 1 \pmod 3$ *– is the single parity-free inequality* $p_D > 1/3$*. Failure mode, and why the even half is genuinely harder: for odd* D *the integrand* Φ_D *of Theorem 3.16 is nonnegative, and Theorem 3.18 follows; for even* D *it is not, and the cancellation is real. Already* $V_2 = 2\big[\Phi_D(2\pi/9) + \Phi_D(4\pi/9) + \Phi_D(8\pi/9)\big]$ *is negative for every even* $D \ge 6$*, because* $\Phi_D(8\pi/9) = -(2\cos(\pi/9))^{D-1}(D - 2\cos(\pi/9))$ *outweighs the positive terms,* $2\cos(\pi/9) > 2\cos(2\pi/9)$*. And no finite sign table of the* W_k *can decide the even half: since* $b_D(L) \sim C\rho_D^L$ *with* $C > 0$*, Corollary 3.17 forces* $W_L \sim C\, 3^{L-1}\rho_D^{L-1}(3\rho_D - f_D)$*, so the eventual sign of* W_k is *the assertion* $\rho_D < f_D/3$ *itself; the signs observed at small* k *(they alternate transiently at base* 3*) are no evidence either way.*

Remark 3.23 (note added, August 2026)*.* The even half of Conjecture 3.22 has since been proved, in every even dimension at once, at base 3 and at base 5: see the companion paper [4], which builds exactly the second idea asked for above – a Collatz–Wielandt certificate closed analytically at depth $\Theta(D^2)$, the transient identified in closed form. The statement above is kept as written, both for the record and because its remaining content, strictness at odd $D \equiv 1 \pmod 3$ beyond the verified range, is still open; [4] reduces it to a single 2-adic valuation bound.

Remark 3.24*.* The size of the excess appears to be governed by the same tower of angles. Numerically,

$$\rho_D - \frac{f_D}{3} \;=\; (-1)^{D+1}\,\frac{2(D-1)}{3}\prod_{k\ge 2}\Big(\cos\frac{2\pi}{3^k}\Big)^{D-1}\frac{D + 2\cos(2\pi/3^k)}{D+2}\cdot(1+o(1)),$$

with the ratio of the two sides within 10^{-8} of 1 at $D = 61$, where the left side is computed by exact rational bisection of χ_{61} against the bracket $\chi_{61}(f_{61}/3) < 0 < \chi_{61}(f_{61}/3+1)$ in `scripts/verify.py`. The per-dimension decay constant is then $\prod_{k\geq 2}\cos(2\pi/3^k) = 0.7428747134\ldots$, and the linear factor $D-1$ is the parity factor of Corollary 3.17. This asymptotic is a conjecture supported by computation; only the inequalities of Theorem 3.18 carry theorem dress.

Conjecture 3.25. *There is no uniform spectral gap: writing λ_2 for a second-largest eigenvalue of $M^{(D)}_{\mathrm{even}}$ in modulus,*

$$\frac{\rho_D}{|\lambda_2|} = \frac{D+2}{D-2} + O(D^{-3}) \longrightarrow 1.$$

Evidence: the computed ratios fall from 3.551785 *at* $D=4$ *to* 1.222222 *at* $D=20$*, monotonically from* $D=6$ *on; over* $6\leq D\leq 20$ *they agree with* $(D+2)/(D-2)$ *to within* 0.05*, and to six decimal places by* $D=18$*. The two smallest dimensions in that range sit outside the agreement,* 3.551785 *against* 3 *at* $D=4$ *and* 2.253050 *against* 2.333333 *at* $D=5$*, which is where the* $O(D^{-3})$ *correction is largest.* `scripts/verify.py` *asserts the monotonicity, and the agreement over* $6\leq D\leq 20$*. Failure mode: the ratios are floating-point roots of exactly computed integer polynomials, so a genuine near-degeneracy at some larger* D *could be misread; and the fitted shape* $(D+2)/(D-2)$ *is an observed match, not a derivation. This is the lighthouse that explains why Theorem 3.18 had to go around the spectrum. The classical route to the sign law reads it off* $\operatorname{sgn}\chi_D(f_D/3) = \prod_i \operatorname{sgn}(f_D/3-\lambda_i)$*, which reports on* ρ_D *alone only if every subdominant eigenvalue sits below* $f_D/3$*; any argument pinning that has to work inside a margin that shrinks like* $4/D$*, and so cannot be a fixed-size perturbation estimate. Four natural routes were tried and refuted by their own computations:* $M^{(D)}_{\mathrm{even}}$ *is not banded plus low rank, not totally nonnegative, not symmetrizable by a positive diagonal, and* $f_D/3\cdot I - M^{(D)}_{\mathrm{even}}$ *is not an M-matrix. The census route of Theorem 3.18 never mentions* λ_2*, which is how the odd half fell; the even half of Conjecture 3.22 still awaits either this separation or a second idea of the same kind.*

4. Proofs

Proof of Lemma 3.2. Split F_D by the number of middle coordinates. A vector with no coordinate equal to 1 chooses 0 or 2 in each of the D places, contributing $(1+t^2)^D$ to the generating function. A vector with exactly one coordinate equal to 1 chooses which coordinate, in D ways, and then 0 or 2 in the remaining $D-1$ places, contributing $Dt(1+t^2)^{D-1}$. No other vector lies in F_D, so

$$P_D(t) = (1+t^2)^D + Dt(1+t^2)^{D-1} = (1+t^2)^{D-1}(1+Dt+t^2).$$

Setting $t=1$ gives $|F_D| = 2^{D-1}(D+2)$. For the coefficients, expand $(1+t^2)^{D-1}(1+t^2) = (1+t^2)^D$ for the even part and $Dt(1+t^2)^{D-1}$ for the odd part: the coefficient of t^{2k} is $\binom{D}{k}$ and that of t^{2k+1} is $D\binom{D-1}{k}$. □

Proof of Theorem 3.1. Step 1: the census is a coefficient. By Definition 2.2 a cell of $S^{(D)}_L$ is exactly a choice of $v^{(0)},\ldots,v^{(L-1)} \in F_D$, and its coordinate sum is $\sum_i x_i = \sum_{j<L} 3^j \sigma_j$ with $\sigma_j = \sum_i v_i^{(j)}$. The number of vectors in F_D with digit sum σ is $B_D(\sigma)$, so for any target T,

$$\#\Big\{x \in S_L^{(D)} : \textstyle\sum_i x_i = T\Big\} = \big[t^T\big] \textstyle\prod_{j=0}^{L-1} P_D(t^{3^j}).$$

Step 2: the carry recursion. For $c \in \mathbb{Z}$ let $N_L(c) = \big[t^{T_L(c)}\big] \prod_{j<L} P_D(t^{3^j})$ with $T_L(c) = D(3^L - 1)/2 + c$, so $a_D(L) = N_L(0)$ and $N_0(c) = \mathbf{1}[c = 0]$. Peel the least significant digit vector, of digit sum s. Since

$$D(3^L - 1)/2 = 3 \cdot D(3^{L-1} - 1)/2 + D,$$

the remaining target after dividing by 3 is $T_{L-1}(c')$ with $c' = (c + D - s)/3$, and the peel is possible exactly when $3 \mid c + D - s$. Therefore

$$N_L(c) = \sum_{\substack{0 \le s \le 2D \\ s \equiv c+D \pmod 3}} B_D(s)\, N_{L-1}\Big(\frac{c + D - s}{3}\Big),$$

which in the notation of Definition 2.4 reads $N_L = (M^{(D)})^L$ applied in the sense $a_D(L) = \big((M^{(D)})^L\big)_{00}$: the entry $M^{(D)}_{c'c} = B_D(c+D-3c')$ is the weight of the step from source c to destination c', and only finitely many entries are nonzero in each row and column because $0 \le c + D - 3c' \le 2D$.

Step 3: contraction. This is Lemma 3.3, proved below. It gives an invariant finite state set, the core C_D, containing the start state 0, so $a_D(L) = \big(M_C^L\big)_{00}$ where M_C is the finite matrix $M^{(D)}$ restricted to C_D.

Step 4: halving. This is Lemma 3.4, proved below. The vector $u_L = M_C^L e_0$ lies, for every L, in the $+1$ eigenspace E of the reflection R inside $\mathbb{R}^{C_D}$, and E is M_C-invariant because R and M_C commute. A basis of E is $v_0 = e_0$ and $v_k = e_k + e_{-k}$ for $1 \le k \le r_D$, so $\dim E = r_D + 1 = \lceil D/2 \rceil = n_D$. Let $M^{(D)}_{\mathrm{even}}$ be the matrix of $M_C|_E$ in that basis and χ_D its characteristic polynomial, of degree n_D. By Cayley–Hamilton, $\chi_D(M_C|_E) = 0$, hence for every $L \ge 0$

$$\sum_{k=0}^{n_D} \chi_{D,k}\, u_{L+n_D-k} = \chi_D(M_C|_E)\, u_L = 0,$$

where $\chi_D(\lambda) = \sum_k \chi_{D,k} \lambda^{n_D - k}$ with $\chi_{D,0} = 1$. Applying the linear functional $w \mapsto w_0$ to this vector identity gives

$$\sum_{k=0}^{n_D} \chi_{D,k}\, a_D(L + n_D - k) = 0,$$

a constant-coefficient linear recurrence of order $n_D = \lceil D/2 \rceil$ satisfied by a_D. The coefficients are integers because $M^{(D)}_{\mathrm{even}}$ has integer entries. □

Proof of Lemma 3.3. If $M^{(D)}_{c'c} \ne 0$ then $c' = (c + D - s)/3$ for some $0 \le s \le 2D$, so $c + D - s$ lies in $[c - D, c + D]$ and

$$|c'| \le \frac{|c| + D}{3}.$$

Suppose $|c| \le r_D = \lfloor (D-1)/2 \rfloor \le (D-1)/2$. Then

$$|c'| \le \frac{(D-1)/2 + D}{3} = \frac{3D-1}{6} = \frac{D}{2} - \frac{1}{6} < \frac{D}{2}.$$

Since c' is an integer, $|c'| < D/2$ forces $|c'| \le \lfloor (D-1)/2 \rfloor = r_D$, whether D is even or odd: at even D an integer below $D/2$ is at most $D/2 - 1 = r_D$, and at odd D it is at most $(D-1)/2 = r_D$. So the core is forward invariant. The start state $c = 0$ lies in the core because $r_D \ge 0$ for $D \ge 2$, and induction on the number of steps finishes the claim. □

Proof of Proposition 3.11. Containment in the core is Lemma 3.3. For the reverse, note first that every coefficient $B_D(s)$ with $0 \le s \le 2D$ is positive, by the closed forms of Lemma 3.2: $B_D(2k) = \binom{D}{k} > 0$ for $0 \le k \le D$ and $B_D(2k+1) = D\binom{D-1}{k} > 0$ for $0 \le k \le D-1$. Hence from carry c the one-step reachable set is exactly the set of integers in $[(c-D)/3,\ (c+D)/3]$, an interval that always contains 0 since $|c| \le r_D < D$. If the states reachable in at most t steps include the interval $[-m, m]$, then one more step reaches every integer in $[-(m+D)/3,\ (m+D)/3]$, because the one-step intervals of consecutive sources overlap. So the reachable radius climbs as $m \mapsto \lfloor (m+D)/3 \rfloor$, starting from $\lfloor D/3 \rfloor$; while $m \le r_D - 1 \le (D-3)/2$ we have $m + D \ge 3m + 3$, so each step gains at least 1, and by Lemma 3.3 the climb stops exactly at r_D. Irreducibility follows since every state reaches 0 in one step and 0 reaches every state; aperiodicity from the loop $M^{(D)}_{00} = B_D(D) > 0$. □

Proof of Lemma 3.4. The involution $v \mapsto (2, \ldots, 2) - v$ of $\{0,1,2\}^D$ preserves the number of coordinates equal to 1, so it permutes F_D and sends digit sum s to $2D - s$. Hence $B_D(s) = B_D(2D - s)$ for all s. Now

$$\begin{aligned} M^{(D)}_{-c',-c} = B_D(-c + D + 3c') &= B_D\big(2D - (c + D - 3c')\big) \\ &= B_D(c + D - 3c') = M^{(D)}_{c'c}, \end{aligned}$$

which is exactly $RM^{(D)}R = M^{(D)}$, and $R^2 = I$ gives $RM^{(D)} = M^{(D)}R$. Clearly $Re_0 = e_0$. Therefore $R\,(M^{(D)})^L e_0 = (M^{(D)})^L Re_0 = (M^{(D)})^L e_0$ for every L, so the whole orbit lies in the $+1$ eigenspace of R. Restricted to the core, that eigenspace consists of the vectors with $w_c = w_{-c}$, spanned by e_0 and $e_k + e_{-k}$ for $1 \le k \le r_D$; these $r_D + 1 = \lceil D/2 \rceil$ vectors are visibly independent. □

Proof of Lemma 3.5. Since $\omega^2 + \omega + 1 = 0$ we have $1 + \omega^2 = -\omega$ and $1 + D\omega + \omega^2 = (D-1)\omega$. Substituting into Lemma 3.2,

$$P_D(\omega) = (-\omega)^{D-1}(D-1)\omega = (-1)^{D-1}\omega^D(D-1),$$

and $|\omega| = 1$ gives $|P_D(\omega)| = D - 1$. □

Proof of Lemma 3.6. Roots of unity filter: for $j \in \{0,1,2\}$,

$$\sum_{s\equiv j} B_D(s) = \frac{1}{3}\sum_{m=0}^{2} \omega^{-jm} P_D(\omega^m) = \frac{f_D}{3} + \frac{2}{3}\operatorname{Re}(\omega^{-j}P_D(\omega)),$$

using $P_D(1) = f_D$ and $P_D(\overline{\omega}) = \overline{P_D(\omega)}$ because P_D has real coefficients. By Lemma 3.5, $\omega^{-j}P_D(\omega) = (-1)^{D-1}(D-1)\,\omega^{D-j}$, whose real part is $(-1)^{D-1}(D-1)$ when $j \equiv D \pmod 3$ and $-(-1)^{D-1}(D-1)/2$ otherwise, since $\operatorname{Re}\omega^k$ is 1 for $k \equiv 0$ and $-1/2$ otherwise. Substituting $\varepsilon_D = (D-1)(-1)^{D-1}/3$ gives $f_D/3 + 2\varepsilon_D$ in the distinguished class and $f_D/3 - \varepsilon_D$ in each of the other two. □

Proof of Proposition 3.7. By Step 1 and Step 2 of the proof of Theorem 3.1 with target $T_L(m)$, the off-centre census is $a_D(L;m) = \big((M^{(D)})^L\big)_{0m} = e_0^\top (M^{(D)})^L e_m$: the source state is the offset m and the accepted final state is 0. By Lemma 3.4, R commutes with $M^{(D)}$, is symmetric, and fixes e_0, so

$$a_D(L;-m) = e_0^\top (M^{(D)})^L R e_m = (Re_0)^\top (M^{(D)})^L e_m = a_D(L;m).$$

Hence $a_D(L;m) = e_0^\top (M^{(D)})^L w$ with $w = \frac{1}{2}(e_m + e_{-m})$. Because $|m| \le r_D$, the vector w lies in the $+1$ eigenspace E of R inside the core, which by Lemma 3.3 and Lemma 3.4 is invariant under M_C and has dimension n_D. So $(M_C)^L w \in E$ for all L, and Cayley–Hamilton applied to $M_C|_E$ annihilates the vector sequence exactly as in Step 4; taking the coordinate functional $w \mapsto w_0$ gives the same recurrence with the same coefficients. □

Proof of Corollary 3.9. By Theorem 3.1, $a_D(L) = \sum_i p_i(L)\lambda_i^L$ where the λ_i are the roots of χ_D and the p_i are polynomials. Let Λ be the largest modulus among the roots that actually occur with a nonzero polynomial; it is positive, because every entry of $M^{(D)}$ is nonnegative and so $a_D(L) = \big((M^{(D)})^L\big)_{00} \ge \big(M_{00}^{(D)}\big)^L = B_D(D)^L > 0$. Summing that expansion, the generating function $\sum_{L\ge 0} a_D(L)z^L$ is a rational function whose poles are exactly the points $1/\lambda_i$ over the occurring nonzero roots, so its radius of convergence is $1/\Lambda$, and Cauchy–Hadamard turns that into $\limsup_L a_D(L)^{1/L} = \Lambda$, the modulus of a root of χ_D. And χ_D is a monic integer polynomial of degree $n_D = \lceil D/2 \rceil$, so every root is an algebraic number of degree at most n_D. If the limit defining $\dim_{\text{slice}}(D)$ exists, then it equals $\log_3 \Lambda$ by taking logarithms. □

Remark 4.1. Exactness of the order is a Hankel condition, not an eigenvalue condition. Given the upper bound n_D of Theorem 3.1, nonvanishing of $H_D = \det\big[a_D(i+j+1)\big]_{0\le i,j<n_D}$ forces the minimal order to be exactly n_D; that is Fact 3.10. Indeed, if a_D obeyed some recurrence $a_D(L+m) = \sum_{k=1}^{m} c_k a_D(L+m-k)$ of order $m < n_D$ for every $L \ge 0$, then for every $i \ge m$ row i of that matrix would be the combination $\sum_{k=1}^{m} c_k \cdot (\text{row } i-k)$ of earlier rows, all n_D rows would lie in the span of the first m, and H_D would vanish. Only this direction holds: a vanishing H_D proves nothing, as the sequence $5, 2, 4, 8, 16, \ldots$ shows, whose minimal order is 2 because $a(L+2) = 2a(L+1)$

for every $L \geq 0$ while no order-one recurrence fits, yet whose shifted Hankel determinant $\det\left(\begin{smallmatrix}2&4\\4&8\end{smallmatrix}\right)$ is zero. Distinct eigenvalues would not suffice either: writing $a_D(L) = \sum_i \lambda_i^L\,(e_0^\top \Pi_i\, e_0)$ with spectral projections Π_i, exactness also needs every weight $e_0^\top \Pi_i e_0$ to be nonzero. The two-line counterexample is $M = \mathrm{diag}(1,2)$ with start vector $e_0 = (1,0)$: the eigenvalues are distinct and the matrix is non-derogatory, yet the sequence $e_0^\top M^L e_0 = 1$ has order 1, not 2. The Hankel determinant sees this where the eigenvalues do not, and it is what `scripts/verify.py` computes, in exact rational arithmetic.

Proof of Lemma 3.15. Write $1 + e^{2i\psi} = 2\cos\psi\, e^{i\psi}$ and $1 + De^{i\psi} + e^{2i\psi} = e^{i\psi}(D + 2\cos\psi)$. Substituting into the factorisation of Lemma 3.2,

$$P_D(e^{i\psi}) = (2\cos\psi)^{D-1} e^{i(D-1)\psi} \cdot e^{i\psi}(D + 2\cos\psi) = e^{iD\psi}\,\Phi_D(\psi).$$

For odd D the power $D-1$ is even, so $(2\cos\psi)^{D-1} \geq 0$, and $D + 2\cos\psi \geq D - 2 > 0$ for $D \geq 3$. For even $D > 2$ the power is odd and $D + 2\cos\psi > 0$, so the sign of Φ_D is the sign of $\cos\psi$. $\square$

Proof of Theorem 3.16. Step 1: the sheaf census is the free-end census. Set $N_p = D(3^p - 1)/2$, an integer for every D because $3^p - 1$ is even. The coordinate sums $T \equiv N_L \pmod{3^L}$ are exactly $T = N_L - 3^L c'$ over $c' \in \mathbb{Z}$, where terms with T outside $[0, D(3^L - 1)]$ vanish, so by Step 1 of the proof of Theorem 3.1,

$$b_D(L) = \sum_{c' \in \mathbb{Z}} [t^{N_L - 3^L c'}] \prod_{j=0}^{L-1} P_D(t^{3^j}).$$

Fix c'. A monomial choice from the product is a tuple $(s_0, \dots, s_{L-1})$ of digit sums with $\sum_j s_j 3^j = N_L - 3^L c'$, weighted by $\prod_j B_D(s_j)$. For $0 \leq p \leq L$ set $T_p = \sum_{j<p} s_j 3^j$ and $c_p = (N_p - T_p)/3^p \in \mathbb{Q}$; then $c_0 = 0$, the identity $3c_{p+1} = c_p + D - s_p$ holds over $\mathbb{Q}$, and $c_p \in \mathbb{Z}$ exactly when $T_p \equiv N_p \pmod{3^p}$. The full congruence $T \equiv N_L \pmod{3^L}$ implies every partial one: $T_p \equiv T \equiv N_L \equiv N_p \pmod{3^p}$, since $N_L - N_p = D(3^L - 3^p)/2 = 3^p N_{L-p}$. So each tuple in the class determines an integer carry path from $c_0 = 0$ with final carry $c_L = c'$, stepping with the weights of Definition 2.4. Conversely, an integer carry path $0 = c_0 \to \cdots \to c_L$ has $s_p = c_p + D - 3c_{p+1}$ and the same telescoping gives $T = N_L - 3^L c_L \equiv N_L \pmod{3^L}$. Hence the weighted count of tuples with final carry c' is $\left((M^{(D)})^L\right)_{c'0}$, paths out of the start state 0 (in Definition 2.4 the first index is the destination). By Lemma 3.3 no path from 0 leaves the core and M_C is a submatrix of $M^{(D)}$ with equal entries, so $b_D(L) = \sum_{c' \in C_D} (M_C^L)_{c'0} = \mathbf{1}^\top M_C^L\, e_0$.

Step 2: extraction. With $\zeta = e^{2\pi i/3^L}$ and $N_L = D(3^L - 1)/2$,

$$b_D(L) = \sum_{T \equiv N_L (3^L)} [t^T] \prod_{j<L} P_D(t^{3^j}) = \frac{1}{3^L} \sum_{m=0}^{3^L - 1} \zeta^{-N_L m} \prod_{j=0}^{L-1} P_D(\zeta^{m 3^j}).$$

Step 3: the phases cancel. Write $\psi_j = 2\pi m 3^j/3^L$. By Lemma 3.15, $P_D(\zeta^{m3^j}) = e^{iD\psi_j}\,\Phi_D(\psi_j)$, and

$$\prod_{j=0}^{L-1} e^{iD\,2\pi m 3^j/3^L} = \exp\Big(2\pi i\, mD\tfrac{3^L-1}{2\cdot 3^L}\Big) = \zeta^{mN_L},$$

which cancels the extraction phase $\zeta^{-N_L m}$ exactly, leaving the stated real sum. It is a positive integer because it equals the census $b_D(L)$, and $b_D(L) \geq a_D(L) \geq B_D(D)^L \geq 1$. □

Proof of Corollary 3.17. Sort the summation index m of Theorem 3.16 by its 3-adic valuation: $m = 0$ contributes $\Phi_D(0)^L = f_D^L$, and $m = 3^{L-k}u$ with $u \in U_k$, $1 \leq k \leq L$, has angles $2\pi u 3^j/3^k$ reduced mod 2π: for $j \geq k$ the angle is an integer multiple of 2π and the factor is $\Phi_D(0) = f_D$; the remaining factors are $\prod_{j<k} \Phi_D(2\pi u 3^j/3^k)$. Hence $3^L b_D(L) = \sum_{k=0}^{L} f_D^{L-k} W_k$, and subtracting f_D times the same identity at $L-1$ leaves $3^L b_D(L) - f_D\, 3^{L-1} b_D(L-1) = W_L$. For the innermost factor, $u3^{k-1}/3^k = u/3$ with $3 \nmid u$, and $2\cos(2\pi/3) = 2\cos(4\pi/3) = -1$ gives $\Phi_D = (-1)^{D-1}(D-1)$ there. □

Proof of Theorem 3.18. The spectral radius is the dominant root. $M^{(D)}_{\text{even}}$ is a square matrix, so its spectral radius is the maximal modulus of the roots of χ_D; that it is itself a root is Perron–Frobenius for nonnegative matrices (no irreducibility required): a nonnegative square matrix has its spectral radius as an eigenvalue, with an entrywise nonnegative eigenvector [8, Theorem 8.3.1]. Nonnegativity of $M^{(D)}_{\text{even}}$ holds because its entries are $B_D(D-3i)$ and $B_D(D+j-3i) + B_D(D-j-3i)$.

Step 1: the census grows at least like $(f_D/3)^L$. For odd D, $\Phi_D \geq 0$ everywhere by Lemma 3.15, so every $W_k \geq 0$, and Corollary 3.17 gives $3^L b_D(L) = \sum_k f_D^{L-k} W_k \geq f_D^L$, i.e. $b_D(L) \geq (f_D/3)^L$ for every L.

Step 2: the growth is bounded by the spectral radius of the core. $b_D(L)$ is the 0-column sum of M_C^L, so $b_D(L) \leq \|M_C^L\|_1$, the maximal column sum. By Gelfand's formula $\|M_C^L\|_1^{1/L} \to \rho(M_C)$, so $\rho(M_C) \geq \limsup_L b_D(L)^{1/L} \geq f_D/3$.

Step 3: the core radius is ρ_D. By Perron–Frobenius again there is $v \geq 0$, $v \neq 0$, with $M_C v = \rho(M_C)\, v$. Its even part $w = (v + Rv)/2$ is nonnegative, and nonzero: $w = 0$ would force $Rv = -v$ with both sides nonnegative, hence $v = 0$. Since R commutes with M_C (Lemma 3.4), w is an eigenvector of M_C at $\rho(M_C)$ lying in the $+1$ eigenspace E, so $\rho(M_C)$ is an eigenvalue of $M^{(D)}_{\text{even}} = M_C|_E$ and $\rho_D \geq \rho(M_C)$. Conversely every eigenvalue of $M_C|_E$ is an eigenvalue of M_C, so $\rho_D \leq \rho(M_C)$. Hence $\rho_D = \rho(M_C) \geq f_D/3$.

Step 4: strictness off the residue class $D \equiv 1$. The matrix $J = f_D I - 3M^{(D)}_{\text{even}}$ has integer entries, and

$$\chi_D(f_D/3) = \det\big((f_D/3)I - M^{(D)}_{\text{even}}\big) = 3^{-n_D} \det J.$$

Mod 3 the term $3M^{(D)}_{\text{even}}$ vanishes, so $\det J \equiv f_D^{n_D} \pmod 3$. Since $f_D = 2^{D-1}(D+2)$, $3 \mid f_D$ exactly when $D \equiv 1 \pmod 3$; for $D \not\equiv 1 \pmod 3$ we get $\det J \not\equiv 0 \pmod 3$, so $\chi_D(f_D/3) \neq 0$ and $f_D/3$ is not an eigenvalue. Since $\rho_D \geq f_D/3$ and ρ_D is an eigenvalue, $\rho_D > f_D/3$. □

Proof of Theorem 3.20. By Lemma 3.3 the core is forward invariant, so no column of M_C loses mass under the restriction: the column of M_C at source c sums to $\sum_{s \equiv c+D \pmod 3} B_D(s)$, which by Lemma 3.6 equals $f_D/3 + 2\varepsilon_D$ when $c \equiv 0 \pmod 3$ and $f_D/3 - \varepsilon_D$ otherwise, with $\varepsilon_D = (D-1)(-1)^{D-1}/3$. The spectral radius of a nonnegative matrix lies between its smallest and largest column sums [8, Theorem 8.1.22], and $\rho(M_C) = \rho_D$ by Step 3 of the proof of Theorem 3.18. Reading off the two values by the parity of ε_D gives the refined brackets, and both are contained in $|\rho_D - f_D/3| \leq 2(D-1)/3$. Finally $2(D-1)/3$ against $f_D/3 = 2^{D-1}(D+2)/3$ gives the relative error $O(D^2 2^{-D})$, and $\log_3$ of $1 + O(D^2 2^{-D})$ tends to 0. □

Proof of Proposition 3.21. By Proposition 3.11, M_C is irreducible, so its Perron vector $v \geq 0$ is unique up to scale and $M_C v = \rho_D v$ with $\rho(M_C) = \rho_D$ as above. Multiply by the all-ones row vector: $\rho_D \sum_c v_c = \mathbf{1}^\top M_C v = \sum_c \sigma(c)\, v_c$, where $\sigma(c)$ is the column sum at c, two-valued as in the proof of Theorem 3.20. Writing $\sigma(c) = f_D/3 + \varepsilon_D\,(3 \cdot \mathbf{1}[3 \mid c] - 1)$ and dividing by $\sum_c v_c$,

$$\rho_D = \frac{f_D}{3} + \varepsilon_D\,(3p_D - 1),$$

which is the identity. Multiplying $\rho_D - f_D/3 = (-1)^{D-1}(D-1)(p_D - \frac{1}{3})$ by $(-1)^{D+1}$ gives the sign statement, and the equivalence with the sign law follows because $(-1)^{D+1}\operatorname{sgn}(\rho_D - f_D/3) = \operatorname{sgn}(p_D - \frac{1}{3})$ at every D. □

5. Reproducibility

Two scripts, plain Python 3, no dependencies, no arguments, every path relative to the lane; run them from the lane root.

`scripts/verify.py` runs in about four seconds and dies loudly on the first mismatch, naming what it got and what it wanted. It covers, in order: the factorisation of Lemma 3.2 and its closed entry form against brute-force enumeration of all 3^D digit tuples for $2 \leq D \leq 8$; agreement of three independent generators, brute enumeration, the substitution product $\prod_{j<L} P_D(t^{3^j})$, and the untrimmed carry automaton on states $-D, \ldots, D$, for $2 \leq D \leq 7$ and $0 \leq L \leq 4$; the exact reachable carry set by breadth-first search for $2 \leq D \leq 24$, corroborating Proposition 3.11; nonvanishing of the exact rational Hankel determinant for $2 \leq D \leq 24$, which is Fact 3.10 and Remark 4.1; the two trace formulas of Fact 3.12 for $2 \leq D \leq 24$; the fill and the three residue-class sums of Lemma 3.6 for $2 \leq D \leq 12$; Proposition 3.7 for $3 \leq D \leq 7$ at every offset $1 \leq m \leq r_D$ over windows through $L = 2n_D + 3$, together with the sharpness witness of Remark 3.8 at $D = 4$, $m = 2$; the anchors of Fact 3.13 at $D = 2, 3, 4, 5, 6$, including the matrix, its characteristic

polynomial, the census, and $\log_3((9+\sqrt{33})/2)$ to six places; the spectral ratios of Conjecture 3.25 for $4 \le D \le 20$, whose monotonicity and agreement with $(D+2)/(D-2)$ are asserted over $6 \le D \le 20$; and both sign checks of the sign law, floating-point roots for $2 \le D \le 20$ and the exact rational sign of $\chi_D(f_D/3)$ for $2 \le D \le 30$, now corroboration of Theorem 3.18 on the odd side and evidence for Conjecture 3.22 on the even side. The new results are covered by five further blocks: the sheaf census of Definition 3.14 computed three independent ways (free-end transfer count, coefficient-class sums of the substitution product in exact integers, and the trigonometric sum of Theorem 3.16 in floating point) for $2 \le D \le 7$; the step identity and parity factor of Corollary 3.17 in exact integers; strict positivity of every W_k, $k \le 8$, at every odd $D \le 25$, the mechanism of Theorem 3.18; nonvanishing of the exact integer $\det(f_D I - 3M^{(D)}_{\text{even}})$ for $2 \le D \le 80$ together with its congruence to $f_D^{n_D}$ mod 3, which is Fact 3.19; and the sign of W_2 at even $6 \le D \le 24$, the obstruction quoted in Conjecture 3.22; and the mass identity of Proposition 3.21 together with both parity-refined brackets of Theorem 3.20, by power iteration on the core matrix for $2 \le D \le 20$. Characteristic polynomials are computed by the Faddeev–LeVerrier recursion over exact rationals; only the spectral checks, the product-formula cross-check and the $D = 3$ dimension use floating point, and the spectral checks do so on integer polynomials rescaled by $f_D/3$ so that all roots have modulus near one.

`scripts/figure.py` runs instantly and redraws `figures/order.svg`, the README's copy of Fig. 1, from the same two formulas $2D+1$ and $\lceil D/2 \rceil$.

The $D = 3$ census is OEIS A299916 [11]. The ladders at $D = 4, 5, 6$ are not in the OEIS to the author's knowledge; no novelty search was run, and they are offered here only as computed values.

Acknowledgments

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